Canonical equivalence relations on nets of PS_c_0

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CANONICAL EQUIVALENCE RELATIONS ON NETS OF PS_{c_0}

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Abstract. We give a list of canonical equivalence relations on discrete nets of the positive unit sphere of c_0. This generalizes results of W. T. Gowers [1] and A. D. Taylor [5].

1. Introduction

Let FIN be the family of nonempty finite sets of positive integers. A block sequence is an infinite sequence $(x_n)_n$ of elements of FIN such that for every n one has $\max x_n < \min x_{n+1}$ (usually written as $x_n < x_{n+1}$). The *combinatorial subspace* $\langle (x_n)_n \rangle$ given by $(x_n)_n$ is the set of finite unions $x_{n_0} \cup \cdots \cup x_{n_m}$. Using this terminology, the Hindman's pigeonhole principle [2] of FIN states that every finite coloring of FIN is constant in some combinatorial subspace, or, equivalently, every equivalent relation on FIN with finitely many classes has a restriction to some combinatorial subspace with only one class. It is easy to see, for example by considering the equivalent relation defined by $s \sim t$ iff $\min s = \min t$, that this is no longer the case for equivalence relations with an arbitrary number of classes. Nevertheless, it is still possible to classify them, much in the spirit of the original motivation of F. P. Ramsey [4] for discovering his famous Theorem. A result of Taylor [5] states that every equivalence relation on FIN can be reduced, by restriction to a combinatorial subspace, to one of the following five canonical relations:

$$\min, \max, (\min, \max), =, \mathrm{FIN}^2,$$

naturally defined by $s \min t$ iff the minimum of s is equal to the minimum of t, $s \max t$ iff the maximum of s is equal to the maximum of t, $s(\min, \max)t$ iff both minimum and maximum are the same.

Following some geometric ideas exposed in Section 2, one can generalize FIN as follows: Given a positive integer k, let FIN_k be the set of mappings $x : \mathbb{N} \to \{0, 1, \dots, k\}$, called *$k$-vectors*, whose support $\operatorname{supp} x = \{n : x(n) \neq 0\}$ is finite and with k in their range. One can naturally extend the union operation on FIN to the join operation $\vee$ on FIN_k by $(x \vee y)(n) = \max\{x(n), y(n)\}$. Let $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ be the mapping defined by $T(x)(n) = \max\{x(n) - 1, 0\}$. A k-block sequence $(x_n)_n$ is an infinite sequence of members of FIN_k such that $\max\operatorname{supp} x_n < \min\operatorname{supp} x_{n+1}$ for every n. The *k-combinatorial subspace* $\langle (x_n)_n \rangle$ defined by a k-block sequence (x_n) is the set of combinations of the form $T^{i_0} x_{n_0} \vee \cdots \vee T^{i_m} x_{n_m}$ with the condition that $i_j = 0$ for some j, and where $T^i x$ is defined by $T^i x(n) = \max\{x(n) - i, 0\}$ for $i > 0$ and $T^0 = Id$. Gowers has proved in [1] that FIN_k possesses the exact analogue of the pigeonhole principle of FIN: Every equivalence relation on FIN_k with finitely many classes has a restriction to some

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combinatorial subspace with only one class. The aim of this paper is to characterize equivalence relations on FIN_k with arbitrary number of classes. More precisely, we are going to give a non redundant finite list $\mathcal{T}_k$ of equivalence relations such that any other equivalence relation on FIN_k can be reduced, modulo restriction to some k-combinatorial subspace, to one in the list $\mathcal{T}_k$.

Indeed, the elements of $\mathcal{T}_k$ are determined by characteristics of a typical k-vector. Easy examples of these are the minimum and maximum of a finite set, that determine the Taylor's list for FIN. Generalizing this, given an integer i with $1 \le i \le k$ let $\min_i s$ be the least integer n such that $s(n) = i$. Another more complex example is the following. Given two integers i and l such that $1 \le l \le i-1 \le k$, let us assign to a given vector s of FIN_k the set of integers n such that $\min_{i-1} s \le n \le \min_i s$ and $s(n) = l$. We illustrate this with the following picture.

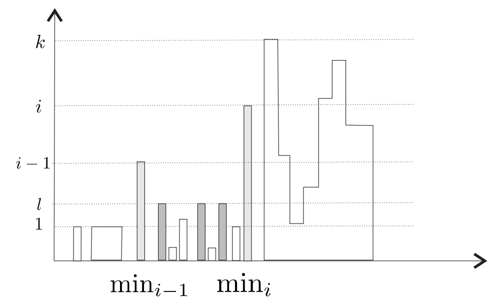

FIGURE 1. an example of invariant

So, our first task will be to guess all the natural characteristics of a k-vector. Although these characteristics are not well defined for an arbitrary k-vector, we will show that every k-block sequence will have a k-*block* subsequence, called here a *system of staircases* for which all the vectors have all natural characteristics well defined. The precise definitions are given in Section 3.

In order to show that every equivalence relation is, when restricted to some k-combinatorial subspace, in $\mathcal{T}_k$ we follow the ideas of Taylor's proof [5]. Let us explain this. Given an equivalence relation $\sim$ on FIN one defines the coloring $c : [\mathrm{FIN}]^{[3]} \to \{0,1\}^4$ by

$$a = (a_0, a_1, a_2) \mapsto \begin{cases} c(a)(0) = 1 & \text{iff } a_0 \sim a_1 \\ c(a)(1) = 1 & \text{iff } a_0 \cup a_1 \sim a_0 \\ c(a)(2) = 1 & \text{iff } a_0 \cup a_1 \sim a_1 \\ c(a)(3) = 1 & \text{iff } a_0 \cup a_1 \cup a_2 \sim a_0 \cup a_2, \end{cases} \tag{1}$$

where $[\mathrm{FIN}]^{[3]}$ is the set of 3-sequences of finite sets (a_0, a_1, a_2) such that $a_0 < a_1 < a_2$. Since $[\mathrm{FIN}]^{[3]}$ has a pigeonhole principle (this is a simple extension of Gowers' result), one can find a block sequence $X = (x_n)_n$ such that c is constant on $[X]^{[3]}$ with value $s_0 \in \{0,1\}^4$. An analysis of the value s_0 identifies the restriction of the equivalence relation $\sim$ to X as one of the five relations $\min, \max, (\min, \max), =, \mathrm{FIN}^2$. Let us re-write the coloring c in a way that will be easy to generalize to FIN_k. Fix an alphabet of countably many variables $\{x_n\}_n$. An $\sim$-equation e is a pair $((x_{i_0}, \dots, x_{i_l}), (x_{j_0}, \dots, x_{j_m}))$, written as $x_{i_0} \cup \cdots \cup x_{i_l} \sim x_{j_0} \cup \cdots \cup x_{j_m}$, such that

$0 = i_0 < \cdots < i_l$, $j_0 < \cdots < j_m$. We say that equation e is true in X iff for every sequence $a_0 < \cdots < a_{\max\{i_l,j_m\}}$ in X the corresponding substitutions $a_{i_0} \cup \cdots \cup a_{i_l}$ and $a_{j_0} \cup \cdots \cup a_{j_m}$ are $\sim$-related; we say that the equation e is false in X iff for every sequence $a_0 < \cdots < a_{\max\{i_l,j_m\}}$ in X one has that $a_{i_0} \cup \cdots \cup a_{i_l} \not\sim a_{j_0} \cup \cdots \cup a_{j_m}$. The equation e is *decided* in X if it is either true or false in X. Using this terminology, one can re-state the fact that the coloring c is constant on X by saying that the equations $x_0 \sim x_1$, $x_0 \cup x_1 \sim x_0$, $x_0 \cup x_1 \sim x_1$ and $x_0 \cup x_1 \cup x_2 \sim x_0 \cup x_2$ are all decided in X. Taylor's proved that these four equations determine the equivalence relation $\sim$. For an arbitrary integer k, the list of equations to be considered is, obviously, longer. For example, for $k = 2$ the equations

$$x_0 \cup x_1 \cup Tx_2 \sim x_1 \cup Tx_2 \text{ and } x_0 + Tx_1 \cup x_2 \sim x_0 \cup x_2,$$

need to be considered. So, the next goal, after one has identified the list $\mathcal{T}_k$, is to find a set L of $\sim$-equations characterizing a given equivalence relation $\sim$ on FIN_k. The first candidate for L is the set of all equations. It turns out that the lists $\mathcal{T}_k$ consists on all the equivalence relations for which every equation is always true or always false, independently of the k-block sequence considered. So it does not seem reasonable to try to find directly a k-block sequence deciding all equations. Instead, we first find a smaller list of equations decided in some k-block sequence, but at the same time large enough to use the inductive hypothesis to provide a richer list of equations, determining our given equivalence relation as one of the list $\mathcal{T}_k$.

It is worth to point out that we give an explicit description of $\mathcal{T}_k$ in a way that it is possible to describe the number t_k of equivalence relations in $\mathcal{T}_k$ using standard arithmetic functions, as for example the incomplete Γ function:

$$t_k = |\mathcal{T}_k| = e^2 \left[k\,[\Gamma(k,1) - \Gamma(k+1,1)]^2 + \Gamma(k+1,1)^2 \right].$$

Since FIN_k is isomorphic to a net of the positive sphere of c_0, our result implies the immediate analogue for those nets. For example, given an equivalence relation R on PS_{c_0} and given some $\delta > 0$ there is an infinite dimensional block subspace X of c_0 and some equivalence relation R' in our finite list such that every R'-class in X is included in the δ-fattening of some R-class.

This paper is organized as follows. In Section 2 we introduce FIN_k as a natural copy of a net of the positive sphere of c_0, extending some standard concepts coming from Banach space theory to FIN_k. We also state the W. T. Gowers Pigeonhole principle of FIN_k. The notion of equation is introduced in Section 3, together with the natural characteristics of a vector of FIN_k. We describe the vectors for which these invariants are well defined, and we show that they appear "everywhere". We also define the family $\mathcal{T}_k$. In Section 4 our main theorem is proved, and in Section 5 we give an explicit formula to compute the cardinality of $\mathcal{T}_k$. Sections 6 and 7 deal with the finite version of our main result, and with some consequences for equivalence relations on the positive sphere of c_0.

2. First Definitions and Results

Recall that $c_0 = c_0(\mathbb{R})$ is the Banach space of sequences of real numbers converging to 0, with the sup-norm defined for a vector $\vec{x} = (x_n)_n$ of c_0 by $\|\vec{x}\| = \sup_n |x_n|$. Let $(e_n)_n$ be its natural Schauder basis, i.e., $e_n(m) = \delta_{n,m}$. The support of a vector $\vec{x} = (x_n)_n$, is defined as supp $\vec{x} = \{n : x_n \neq 0\}$ and let c_{00} be the linear subspace of c_0 consisting of the vectors $\vec{x} = (x_n)_n$ with finite support, i.e., only finitely many of the coordinates of $\vec{x}$ are not zero. Given two vectors $\vec{x}$ and $\vec{y}$ of c_{00} we write $\vec{x} < \vec{y}$ to denote that $\max \operatorname{supp} \vec{x} < \min \operatorname{supp} \vec{y}$.

Let PS_{c_0} be the set of norm one positive vectors of c_0, i.e., the set of all vectors $\vec{x} = (x_n)_n$ such that $\|\vec{x}\| = 1$, and such that $x_n \geq 0$, for every n, and let PB_{c_0} be the set of positive vectors of the unit ball of c_0. Observe that PB_{c_0} is a lattice with respect to $(x_n)_n \vee (y_n)_n = (\max\{x_n, y_n\})_n$ and $(x_n)_n \wedge (y_n)_n = (\min\{x_n, y_n\})_n$, with $0 = (0)_n$, and $1 = (1)_n$. Notice also that PS_{c_0} is closed under the operation $\vee$, and that $x \vee y = x + y$ if x and y have disjoint support. In general, given two subsets $N \subseteq A$ of c_0 and a positive number δ we say that N is a *δ-net* of A iff for every $\vec{a} \in A$ there is some $\vec{x} \in N$ such that $\|\vec{a} - \vec{x}\| \leq \delta$.

For a given δ with $0 < \delta < 1$, let k be the least integer such that $1/(1+\delta)^{k-1} \leq \delta$, and let $\varepsilon = 1/(1+\delta)$. Let

$$\begin{aligned} \mathcal{N}_\delta =& \{x \in PB_{c_{00}} : x(i) \in \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{k-1}, 0\}\} \\ \mathcal{M}_\delta =& \{x \in PS_{c_{00}} : x(i) \in \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{k-1}, 0\}\}. \end{aligned}$$

Since $\varepsilon^i - \varepsilon^{i+1} = \varepsilon^i(1-\varepsilon) = \varepsilon^i(\delta/(1+\delta)) < \delta$ and $\varepsilon^{k-1} \leq \delta$, it follows that $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are δ-nets of PB_{c_0} and of PS_{c_0}, respectively. The set $\mathcal{N}_\delta$ is a sub-lattice of PB_{c_0} with respect to $\vee$ and $\wedge$, and it is closed under scalar multiplication by ε, identifying $\varepsilon^l = 0$ for $l \geq k$ (which means that we identify the coordinates less than ε^k with 0). Also, for two $\vec{x}, \vec{y} \in \mathcal{M}_\delta$, we have that $\vec{x} \vee \varepsilon^i \vec{y}, \varepsilon^i \vec{x} \vee \vec{y} \in \mathcal{M}_\delta$, for every $0 \leq i \leq k-1$. Finally, note that $\mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \varepsilon^i \mathcal{M}_\delta$, a disjoint union.

We define the mapping $\Theta = \Theta_\delta : \mathcal{N}_\delta \to \{0, 1, \dots, k\}^{\mathbb{N}}$ by

$$\Theta((x_m)_m)(n) = \begin{cases} k - \log_\varepsilon(x_n) & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0. \end{cases}$$

We can now give an equivalent definition of FIN_k using the mapping Θ, and therefore giving a geometrical interpretation of it.

Definition 2.1. Fix, for a given integer k, a positive real number $\delta = \delta(k)$ such that $1/(1+\delta)^{k-1} = \delta$. Let $\mathrm{FIN}_k = \Theta(\mathcal{M}_\delta)$, i.e., the set of functions $s : \mathbb{N} \to \{0, 1, \dots, k\}$ eventually 0, and with k in the range. The elements of FIN_k are called *k-vectors*.

Observe that $\Theta"\mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \Theta"\varepsilon^i \mathcal{M}_\delta$, and that $\Theta"\varepsilon^i \mathcal{M}_\delta$ is the set of all functions $s : \mathbb{N} \to \{0, 1, \dots, k\}$ eventually 0, and with $k-i$ in the range. So, $\Theta"\varepsilon^i \mathcal{M}_\delta = \mathrm{FIN}_{k-i}$. Hence, $\Theta"\mathcal{N}_\delta = \bigcup_{i=1}^{k} \mathrm{FIN}_i =: \mathrm{FIN}_{\leq k}$, whose members are called *$(\leq k)$-vectors*.

We can transfer the algebraic structure of $\mathcal{N}_k \subseteq c_{00}$ to $\mathrm{FIN}_{\leq k}$ via Θ. In particular, for $s, t \in \mathrm{FIN}_{\leq k}$, let the *support of* s be $\operatorname{supp} s = \{n : s(n) \neq 0\}$; we write $s < t$ to denote that $\max \operatorname{supp} s < \min \operatorname{supp} t$, define $s \vee t$ and $s \wedge t$ by

$$(s \vee t)(n) = \max\{s(n), t(n)\} \text{ and } (s \wedge t)(n) = \min\{s(n), t(n)\},$$

and let T be the transfer of the multiplication by ε, i.e., for a $(\leq k)$-vector s, let

$$T(s) = \Theta(\varepsilon \Theta^{-1}(s)) = (s-1) \vee 0.$$

Let $S : \mathrm{FIN}_{k-1} \to \mathrm{FIN}_k$ be an inverse map for T, defined for a $(k-1)$-vector a by

$$S(a)(n) = \begin{cases} a(n)+1 & \text{if } n \in \operatorname{supp} a \\ 0 & \text{if not.} \end{cases}$$

It turns out that $\mathrm{FIN}_{\leq k}$ is a lattice with operations $\vee$ and $\wedge$, and it is closed under T. We will use the order $\leq_L$ to denote the lattice-order of $\mathrm{FIN}_{\leq k}$, i.e., for $s, t \in \mathrm{FIN}_{\leq k}$, we write $s \leq_L t$ iff $s \wedge t = s$. Note that $\mathrm{FIN}_i \vee \mathrm{FIN}_j = \mathrm{FIN}_{\max\{i,j\}}$ and $\mathrm{FIN}_i \wedge \mathrm{FIN}_j = \mathrm{FIN}_{\min\{i,j\}}$. We will use $s+t$ for $s \vee t$ whenever $s < t$.

We now pass to introduce some combinatorial notions. A sequence of k-vectors (s_n) is called a *finite k-block sequence* if (s_n) is finite and if $s_n < s_{n+1}$ for every n; if such sequence is infinite, then we call it a (infinite) *k-block sequence*. We write $\mathrm{FIN}_k^{[\infty]}$, $\mathrm{FIN}_k^{[n]}$ and $\mathrm{FIN}_k^{[<\infty]}$ to denote respectively the set of k-block sequences, finite k-block sequences of length n, and the set of finite k-block sequences.

The *k-combinatorial subspace* $\langle \alpha \rangle$ defined by a finite or infinite k-block sequence $\alpha = (s_n)_n$ is the set of all k-vectors of α defined by

$$\langle \alpha \rangle = \Theta((\mathrm{LinSpan}\ \Theta^{-1}\{s_n\}_n) \cap \mathcal{N}_\delta),$$

where LinSpan A denotes the linear span of a given subset A of c_0. Using this one has that $\mathrm{FIN}_k = \langle (\Theta e_n)_n \rangle$. Similarly, we define for a given integer $i \leq k$ the set $\langle \alpha \rangle_i$ of i-vectors of α. A main property of the k-block sequences $(a_n)_n$ is that $e_n \mapsto a_n$ naturally extends to a lattice isomorphism between FIN_k and $\langle (a_n)_n \rangle$ that preserves the operation T.

For $M \leq N \leq \infty$, and $\alpha = (s_n)_{n<N}$ let $[\alpha]^{[M]}$ be the set of *k-block subsequences* of α, defined as $[\alpha]^{[M]} = \{(s_n)_{n<M} \in \mathrm{FIN}_k^{[M]} \ : \ s_n \in \langle \alpha \rangle \, (0 \leq n < M)\}$. Without loss of generality we will identify $[\alpha]^{[1]}$ with $\langle \alpha \rangle$.

Given two finite block sequences α and β, and two infinite ones A and B, we define $\alpha \preceq \beta$ if and only if $\alpha \in [\beta]^{[|\alpha|]}$, $\alpha \preceq A$ if and only if $\alpha \in [A]^{[|\alpha|]}$, and $B \preceq A$ if and only if $B \in [A]^{[\infty]}$. Notice that all these definitions come from the notion of subspace. For example, $A \in \langle B \rangle$ if and only if the space generated by $\Theta^{-1}A$ is a subspace of the space generated by $\Theta^{-1}B$.

For a k-block sequence $A = (a_i)_i$ and $a \in \langle A \rangle$, since $\langle A \rangle = \Theta(\mathrm{LinSpan}\ \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_\delta)$, we have that $\Theta^{-1}a \in \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_\delta$. Therefore, $\Theta^{-1}a = \sum_{i=0}^m \varepsilon^{d_i}\Theta^{-1}a_i$, for some m, and with possibly some $d_i = 0$. This implies that $a = \Theta(\sum_{i=0}^m \varepsilon^{d_i}\Theta^{-1}a_i) = \sum_{i=0}^m \Theta(\varepsilon^{d_i}\Theta^{-1}a_i) = \sum_{i=0}^m T^{d_i}a_i$.

Finally, an infinite sequence $(A_r)_{r\in\mathbb{N}}$ of infinite k-block sequences $A_r = (a_n^r)_n$ is called a *fusion sequence of* $A \in \mathrm{FIN}_k^{[\infty]}$ if for all $r \in \mathbb{N}$:
(a) $A_{r+1} \preceq A_r \preceq A$,
(b) $a_0^r < a_0^{r+1}$.

The infinite k-block sequence $A_\infty = (a_0^r)_{r\in\mathbb{N}}$ is called the *fusion k-block sequence* of the sequence $(M_r)_{r\in\mathbb{N}}$.

Definition 2.2. Given a k-block sequence $A = (a_n)_n$, let $C_A : \langle A \rangle \to FIN_k$ be the mapping satisfying

$$a = \sum_{n=0}^{\infty} T^{k - C_A(a)(n)} a_n, \tag{2}$$

for every k-block vector a of A. Since $\Theta^{-1}a=\sum_{n\geq 0}\varepsilon^{k-C_A(a)(n)}\Theta^{-1}a_n$, for every a, the mapping C_A is well defined. We call the sum in (2) the *canonical decomposition of a in A*. Notice that $C_A(a)\in \mathrm{FIN}_k$ for every a.

For two $(\leq k)$-vectors s and t,
(a) we write $s\sqsubseteq t$ when $t\restriction \mathrm{supp}\ s=s$, i.e., if t restricted to the support of s is equal to s, and
(b) we write $s\perp t$ when there is no $u\in \mathrm{FIN}_{\leq k}$ such that $u\sqsubseteq s,t$, i.e., if $s(n)\neq t(n)$ for every $n\in \mathrm{dom}\ s\cap \mathrm{dom}\ t$.

Using this, if $s=\sum_{n=0}^{\infty}T^{k-l_n}a_n$, then $T^{k-l_n}a_n\sqsubseteq a$, for every n, while $T^{k-l_n}a_n\perp T^{k-l_{n'}}a_{n'}$ for every $n\neq n'$. It follows that:

Proposition 2.3. *Fix $A=(a_n)_n$, $a\in\langle A\rangle$ and an integer n. If there are some $r\leq k$ and m such that $T^{k-r}a_n(m)=a(m)\neq 0$, then necessarily $C_A(a)(n)=r$ (i.e., $T^{k-r}a_n\sqsubseteq a$).* □

The following is Gowers' pigeonhole principle for FIN_k.

Theorem 2.4. [1] *If FIN_k is partitioned into finitely many pieces, then there is $A\in \mathrm{FIN}_k^{[\infty]}$ such that $\langle A\rangle$ is in only one of the pieces.*

This naturally extends to higher dimensions.

Lemma 2.5. [6] *Suppose that $f:\mathrm{FIN}_k^{[n]}\to\{0,\dots,l-1\}$. Then there is a block sequence X such that f is constant on $[X]^{[n]}$.*

Proof. The proof is done by induction on n. Suppose it is true for $n-1$. We can find, by a repeated use of Theorem 2.4, a fusion sequence $(X_r)_r$, $X_r=(x_i^r)_i$, such that for every r and every $(b_0,\dots,b_{n-2})\in[(x_i^i)_{i<r}]^{[n-1]}$ the coloring f is constant on the set $\{(b_0,\dots,b_{n-2},x)\,:\,x\in X_r\}$ with value $\varepsilon((b_0,\dots,b_{n-2}),r)$. By construction one has that $X_r\preceq X_s$ if $r\leq s$. So it follows that $\varepsilon((b_0,\dots,b_{n-2}),r)=\varepsilon((b_0,\dots,b_{n-2}),s)$ for every $(b_0,\dots,b_{n-2})\in[\theta_r]^{[n-1]}$ and every $r<s$. This allows us to define $\varepsilon:[X_\infty]^{[n-1]}\to\{0,1,\dots,l-1\}$ by $\varepsilon(b_0,\dots,b_{n-2})=\varepsilon((b_0,\dots,b_{n-2}),r)$, for some (any) integer r, where $X_\infty=(x_i^i)_i$ is the fusion k-block sequence of $(X_r)_r$. This coloring ε can be easily interpreted as a coloring of $\mathrm{FIN}_k^{[n-1]}$, so by the inductive hypothesis there is some $X\preceq X_\infty$ such that ε is constant on $[X]^{[n-1]}$, and therefore f is also constant on $[X]^{[n]}$. □

3. Equations, Staircases and Canonical equivalence Relations

Roughly speaking, terms are natural mappings that assign k-vectors to finite block sequences of k-vectors of a fixed length n, and which are defined from the operations $+$ and T^i of FIN_k. For example, the mapping that assigns to a block sequence (a_1,a_2) of k-vectors the k-vector a_1+Ta_2 is a k-term which can be understood as the mapping with two variables x_1,x_2 defined by $f(x_1,x_2)=x_1+Tx_2$.

From two fixed k-terms f and g of n variables and one equivalence relation $\sim$ on FIN_k we can define the natural coloring $c_{f,g}:[\mathrm{FIN}_k]^{[n]}\to\{0,1\}$ via $c_{f,g}(a_1,\dots,a_n)=1$ if and only if $f(a_1,\dots,a_n)\sim g(a_1,\dots,a_n)$. A k-equation will be $f\sim g$. The pigeonhole principle in Lemma 2.5 gives that for every equation $f\sim g$ (f and g with n variables) there is some infinite block sequence A such that, either for every $(a_1,\dots,a_n)$ in $[A]^{[n]}$, $f(a_1,\dots,a_n)\sim g(a_1,\dots,a_n)$, or for all $(a_1,\dots,a_n)$ in $[A]^{[n]}$, $f(a_1,\dots,a_n)\not\sim g(a_1,\dots,a_n)$, i.e., in A the equation $f\sim g$ is either true or false. As we explained in the introduction, Taylor proves that an equivalence relation $\sim$ on FIN is determined by a list of 4 equations (precisely, $x_0 \sim x_1$, $x_0 \sim x_0 + x_1$, $x_1 \sim x_0 + x_1$ and $x_0 + x_1 + x_2 \sim x_0 + x_2$). This is going to be also the case for arbitrary k, of course with a more complex list of equations.

3.1. **Terms and equations.**

Definition 3.1. Let $\mathtt{X} = \{x_n\}_{n\geq 1}$ be a countable infinite alphabet of variables. Consider the trivial map $\mathtt{x} : \mathtt{X} \to \mathbb{N}$, defined by $x_n \mapsto \mathtt{x}(x_n) = n$. A *free k-term* $\mathtt{p}$ is a map of the form $s \circ \mathtt{x}$ where s is a k-vector, i.e., it is a map $\mathtt{p} : \mathtt{X} \to \{0, \dots, k\}$ such that supp $\mathtt{p}$ is finite, and k is in the range of $\mathtt{p}$. A natural representation of $\mathtt{p}$ is

$$\mathtt{p} = \mathtt{p}(x_0, \dots, x_l) = \sum_{i=0}^{l} T^{k-m_i} x_i,$$

where $0 \leq m_i \leq k$, and at least one $m_i = k$. For example $T^2 x_1 + T x_2 + x_4$, and $x_1 + x_5$ are both free 3-terms. Notice that, if $\mathtt{p}$ is a free k-term, then $\mathtt{p} \circ \mathtt{x}^{-1}$ is a k-vector. A *free $(\leq k)$-term* is $s \circ \mathtt{x}$, where s is a $(\leq k)$-vector. It follows that the set of free $(\leq k)$-terms is a lattice. For example

$$\mathtt{p}(x_0, \dots, x_n) \vee \mathtt{q}(x_0, \dots, x_m) = (\mathtt{p} \circ \mathtt{x}^{-1} \vee \mathtt{q} \circ \mathtt{x}^{-1}) \circ \mathtt{x}.$$

We also have defined the operator T for a k-term $p(x_0, \dots, x_n)$ by

$$T(p(x_0, \dots, x_n)) = (T(p \circ \mathtt{x}^{-1}) \circ \mathtt{x}).$$

For every $(\leq k)$-term $p(x_0, \dots, x_n) = \sum_{i=0}^{n} T^{k-m_i} x_i$ we consider the following kind of substitutions:

(a) Given a sequence of free $(\leq k)$-terms $t_0, \dots, t_n$, consider the substitution of each x_i by t_i

$$\mathtt{p}(t_0, \dots, t_n) = \bigvee_{i=0}^{n} T^{k-m_i} t_i.$$

In the case that $\mathtt{p}$ and $t_0, \dots, t_n$ are free k-terms, then $\mathtt{p}(t_0, \dots, t_n)$ is also a free k-term.

(b) For a block sequence $(a_0, \dots, a_n)$ of $(\leq k)$-vectors, replace each x_i by a_i

$$\mathtt{p}(a_0, \dots, a_n) = \sum_{i=0}^{n} T^{k-m_i} a_i.$$

If $\mathtt{p}$ is a free k-term, and $a_0, \dots, a_n$ are k-vectors, then the result of the substitution $\mathtt{p}(a_0, \dots, a_n)$ is a k-vector. The main reason to introduce free k-terms is the following notion of equations.

Definition 3.2. A *free k-equation* (*free equation* in short) is a pair $\{\mathtt{p}(x_0, \dots, x_n), \mathtt{q}(x_0, \dots, x_{n'})\}$ of free k-terms. Given a fixed equivalence relation $\sim$ on FIN_k, we will write the previous free equation as

$$\mathtt{p}(x_0, \dots, x_n) \sim \mathtt{q}(x_0, \dots, x_{n'}).$$

Given s, t, i_0 and i_1-vectors respectively, a free j_0-term $\mathtt{p}$, and a free j_1-term $\mathtt{q}$ such that $\max\{i_l, j_l\} = k$ for $l = 0, 1$, we consider the equations of the form $s + \mathtt{p} \sim t + \mathtt{q}$ and $\mathtt{p} + s \sim \mathtt{q} + t$, called *$k$-equations* (or *equations*, if there is no possible confusion). The substitutions of $(b_0, \dots, b_n)$ in the equation $s + p \sim t + q$ will be allowed only when $b_0 > s, t$, and for an equation $\mathtt{p} + s \sim \mathtt{q} + t$, provided that $b_n < s, t$. This last condition implies that only finitely many substitutions are allowed for this latter equations, in contrast with the equations of the form $s + p \sim t + q$.

Definition 3.3. We say that a k-equation $s+p(x_0,\dots,x_n) \sim t+q(x_0,\dots,x_n)$ (or $p(x_0,\dots,x_n) + s \sim q(x_0,\dots,x_n)+t$) *holds* (or is *true*) in A iff for every $(a_0,\dots,a_n)$ in $[A]^{[n+1]}$ with $a_0 > s,t$ (resp. $a_n < s,t$), $s+p(a_0,\dots,a_n) \sim s+q(a_0,\dots,a_n)$ (resp. $p(a_0,\dots,a_n)+s \sim q(a_0,\dots,a_n)+s$). The equation $s+p(x_0,\dots,x_n) \sim t+q(x_0,\dots,x_n)$ (or $p(x_0,\dots,x_n)+s \sim q(x_0,\dots,x_n)+t$) is false in A iff for every $(a_0,\dots,a_n)$ in $[A]^{[n+1]}$ with $a_0 > s,t$ (resp. $a_n < s,t$), $s+p(a_0,\dots,a_n) \not\sim s+q(a_0,\dots,a_n)$ (resp. $p(a_0,\dots,a_n)+s \not\sim q(a_0,\dots,a_n)+s$). The equation is *decided in A* iff it is either true in A or false in A.

It is clear that, given a k-equation $\mathtt{p}(x_0,\dots,x_n) \sim \mathtt{q}(x_0,\dots,x_{n'})$, we can assume that $n = n'$, since we can extend the terms of the equation adding summands of the form $T^k x$ and not changing the “meaning” of the k-equation.

Some properties of equations that will be useful are given in the following.

Proposition 3.4. *Suppose that all free k-equations with at most five variables are decided in a given k-block sequence A. Then:*

(i) *If $x_0 + T^{k-i}x_1 + x_2 \sim x_0 + x_2$ is true in A, then $x_0 + T^{k-j}x_1 + x_2 \sim x_0 + x_2$ is true in A for every $j \le i$.*

(ii) *If $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ or $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ are true in A, then $x_0 + x_1 + x_2 \sim x_0 + x_2$ is also true in A.*

(iii) *If the equation $x_0 + x_1 + T^i x_2 \sim x_0 + T^i x_2$ is true in A, then the equation $x_0 + x_1 + T^j x_2 \sim x_0 + T^j x_2$ also is true in A for every $j \le i$.*

(iv) *If the equation $T^i x_0 + x_1 + x_2 \sim T^i x_0 + x_2$ is true in A, then the equation $T^j x_0 + x_1 + x_2 \sim T^j x_0 + x_2$ also is true in A for every $j \le i$.*

(v) *If the equation $x_0 + T^{k-r_1}x_1 + T^{k-r_0}x_2 \sim x_0 + T^{k-r_0}x_2$ holds, then also the equation $x_0 + T^{k-r_2}x_1 + T^{k-r_0}x_2 \sim x_0 + T^{k-r_0}x_2$ for every $r_1 > r_2$ and r_0.*

Proof. Suppose that the k-block sequence A decides all the equations with at most five variables.

(i): Fix $j < i$. Then,

$$x_0 + T^{k-i}x_1 + T^{k-j}x_2 + x_3 \sim x_0 + T^{k-i}(x_1 + T^{i-j}x_2) + x_3 \sim x_0 + x_3 \text{ hold in } A. \tag{3}$$

Hence,

$$x_0 + T^{k-i}x_1 + (T^{k-j}x_2 + x_3) \sim x_0 + (T^{k-j}x_2 + x_3) \text{ holds in } A, \tag{4}$$

and we are done.

(ii): Suppose now that $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is true in A. Then

$$x_0 + x_2 + Tx_3 \sim x_0 + Tx_3 \text{ and } x_0 + x_1 + x_2 + Tx_3 \sim x_0 + Tx_3 \text{ are true in } A. \tag{5}$$

Hence, $x_0 + x_1 + x_2 + Tx_3 \sim x_0 + x_2 + Tx_3$ holds in A, and therefore, $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true in A.

(iii): Suppose that $x_0 + x_1 + T^i x_2 \sim x_0 + T^i x_2$ is true in A, and fix $j \geq i$. Then, $x_0 + x_1 + x_2 + T^j(x_3 + T^{i-j}x_4) \sim x_0 + x_1 + x_2 + T^j x_3 + T^i x_4 \sim x_0 + T^i x_4$ hold in A, and

$$x_0 + x_1 + T^j(x_2 + T^{i-j}x_3) \sim x_0 + x_1 + T^j x_2 + T^i x_3 \sim x_0 + T^i x_3 \text{ hold in } A, \tag{6}$$

which implies what we wanted.

(iv): This is showed in a similar manner that (iii).

(v): Fix $r_1 > r_2$ and r_0 and suppose that the equation $x_0 + T^{k-r_1}x_1 + T^{k-r_0}x_2 \sim x_0 + T^{k-r_0}x_2$ holds in A. Then, $x_0 + T^{k-r_2}x_1 + T^{k-r_1}x_2 + T^{k-r_0}x_3 \sim x_0 + T^{k-r_1}(T^{r_1-r_2}x_1 + x_2) + T^{k-r_0}x_3 \sim x_0 + T^{k-r_0}x_3$ and $(x_0 + T^{k-r_2}x_1) + T^{k-r_1}x_2 + T^{k-r_0}x_3 \sim x_0 + T^{k-r_2}x_1 \sim T^{k-r_0}x_3$ holds in A. Therefore, $x_0 + T^{k-r_2}x_1 \sim T^{k-r_0}x_3 \sim x_0 + T^{k-r_0}x_3$ is true in A. □

3.2. **Systems of staircases, canonical and staircase equivalence relations.** Classifying equivalence relations of FIN_k is roughly the same as finding properties of a typical k-vector. One of these properties can be the cardinality, or, for example, the minimum or maximum of its support. Indeed Taylor's result on FIN tells that these are the relevant properties of 1-vectors. For an arbitrary $k > 1$, one expects a longer list of properties. One example is obtained by considering for a given k-vector a the least integer n of the support of a such that $a(n) = k$; another one is obtained by fixing i with $1 \leq i \leq k$ and considering the least n such that $a(n) = i$. This is not always well defined, since for $i < k$ there are k-vectors where i does not appear in their range. Nevertheless, this last property seems very natural to consider. Indeed we are going introduce a type of k-block sequences, called systems of staircases, where these properties, and some others, are well defined for every k-vector of their combinatorial subspaces.

Definition 3.5. Given an integer $i \in [1, k]$ let $\min_i, \max_i : FIN_k \to \mathbb{N}$ be the mappings $\min_i(s) = \min s^{-1}\{i\}$, $\max_i(s) = \max s^{-1}\{i\}$, if defined, and 0 otherwise. A k-vector a is a *system of staircases* (*sos* in short) if and only if

(i) Range $s = \{0, 1 \ldots, k\}$,
(ii) $\min_i a < \min_j a < \max_j a < \max_i a$, for $i < j \leq k$,
(iii) for every $1 \leq i \leq k$,

$$\text{Range } a\restriction[\min_{i-1}a, \min_i a] = \{0, \ldots, i\},$$
$$\text{Range } a\restriction[\max_i a, \max_{i-1}a] = \{0, \ldots, i\},$$
$$\text{Range } a\restriction[\min_k a, \max_k a] = \{0, \ldots, k\}.$$

The following figure illustrates the previous definition.

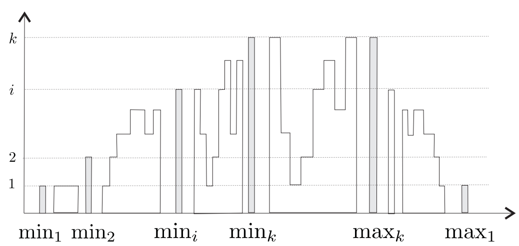

FIGURE 2. A typical sos.

A block subspace $A = (a_n)_n$ is a *system of staircases* iff every k-vector in $\langle A\rangle$ is an sos. In the next proposition we show, among other properties, that for every k-block sequence A there is sos $B \in [A]^{[\infty]}$.

Proposition 3.6.
(i) *T preserves sos, i.e., if a is an sos k-vector, then Ta is an sos $(k-1)$-vector.*
(ii) *$T^{k-j}a+b$, $a+T^{k-j}b$ are sos's, provided that $a < b$ are sos's. Therefore, for every k-term $p(x_0, \dots, x_n)$ and every block sequence of sos $(a_0, \dots, a_n) \in [\mathrm{FIN}_k]^{[n+1]}$, the substitution $p(a_0, \dots, a_n)$ is also an sos.*
(iii) *A k-block sequence $A = (a_n)_n$ is an sos if and only if a_n is an sos for every n.*
(iv) *If A is an sos, then any other $B \preceq A$ is also an sos.*
(v) *For every A there is some $B \preceq A$ which is an sos.*

Proof. It is not difficult to prove (i) and (ii) (for the last part of (ii), one can use induction on the complexity of the k-term p). To show (iii), let us suppose that a_n is an sos for every n, and let us fix $a \in \langle (a_n)_n\rangle$. Then there is a k-term $p(x_0, \dots, x_n)$ such that $p(a_0, \dots, a_n) = a$. Therefore, by (ii), a is an sos. Assertion (iv) easily follows from (ii). Finally, Let us prove (v): Fix $A = (a_n)_n$. For each n, let

$$c_n = \sum_{j=1}^{k} T^{k-j} a_{(2k-1)n+j-1} + \sum_{j=1}^{k-1} T^{k-(k-j)} a_{(2k-1)n+k-1+j}.$$

Notice that for every n one has that

$$\text{Range } c_n\restriction[0, \min_k(c_n)] = \text{Range } c_n\restriction[\max_k(c_n), \infty) = \{0, \dots, k\}.$$

Therefore, Range $T^{k-j}c_n\restriction[0, \min_j T^{k-j}(c_n)]$ = Range $T^{k-j}c_n\restriction[\max_j T^{k-j}(c_n), \infty) = \{0, \dots, j\}$ for each $j \le k$. For $n \ge 0$, let

$$b_n = \sum_{j=1}^{k} T^{k-j} c_{n(3k-1)+j-1} + \sum_{j=1}^{k} T^{k-j} c_{n(3k-1)+k-1+j} + \sum_{j=1}^{k-1} T^{k-(k-j)} c_{n(3k-1)+2k-1+j}.$$

Now it is not difficult to prove that every b_n is an sos. □

Definition 3.7. An equivalence relation $\sim$ on FIN_k is *canonical*[1] *in* A if and only if every k-equation are decided in every sos $B \in [A]$ in the same way, i.e., iff for every k-equation $p \sim q$, either for every sos $B \in [A]$ one has that $p \sim q$ is true in B, or for every sos $B \in [A]$ one has that $p \sim q$ is false in B. We will say that $\sim$ is canonical if it is canonical in FIN_k.

Canonical equivalence relations are those for which all the equations $p \sim q$ are decided in every sos in the same way. It is not difficult to see that all the equivalence relations of the list $\{\min, \max, (\min, \max), =, \mathrm{FIN}^2\}$ are canonical in FIN. Taylor's result for FIN says that there are no more canonical equivalence relations than the ones in this list. It will be shown later that

[1]this name is not arbitrary chosen: We will show that every equivalence relation is, when restricted to some combinatorial subpace, canonical.

for every k there is also a finite list of canonical equivalence relations. Indeed we will give an explicit description of how canonical equivalence relations look like.

In order to do the same to the equivalence relations in FIN_k we have to give a list of relations naturally defined for a typical sos.

Definition 3.8. For a set X, a k-block sequence A, and an arbitrary map $f : \langle A\rangle \to X$ we define the relation R_f on $\langle A\rangle$ by $sR_f t$ if and only if $f(s) = f(t)$. Whenever there is no possible confusion, we are going to use the notation sft instead of $sR_f t$. Now fix an sos A. Recall that $\min_i(s) = \min\{n : s(n) = i\}$ for a given integer $i \in [1, k]$ and $s \in \langle A\rangle$. This mapping can be interpreted as $\min_i : \langle A\rangle \to FIN_i$ in the following way

$$\min_i(s)(n) = \begin{cases} i & \text{if } n = \min_i(s) \\ 0 & \text{otherwise.} \end{cases}$$

Extending this, define, for $I \subseteq \{1, \dots, k\}$, the mapping $\min_I : \langle A\rangle \to \mathrm{FIN}_{\max I} \subseteq \mathrm{FIN}_{\leq k}$ by $\min_I(s)(n) = i$ if $n = \min_i(s)$, for $i \in I$ and 0 otherwise, i.e., $\min_I(s) = \{(\min_i(s), i) : i \in I\}$, and extended by 0 in the rest. Similarly, let

$$\max_i(s)(n) = \begin{cases} i & \text{if } n = \max_i(s) \\ 0 & \text{otherwise,} \end{cases}$$

and let $\max_I : FIN_k \to FIN_{\max I}$ be defined by $\max_I(s) = \{(\max_i(s), i) : i \in I\}$, again extended by 0. Clearly $\min_I = \bigvee_{i\in I} \min_i$ and $\max_I = \bigvee_{i\in I} \max_i$, where for two mappings $f, g : \langle A\rangle \to \mathrm{FIN}_{\leq k}$ we define $(f \vee g)(s) = f(s) \vee g(s)$.

We now introduce a more sophisticated class of functions. For $l \leq i-1$, let $\theta^0_{i,l}, \theta^1_{i,l} : \langle A\rangle \to FIN_l$ be the mappings defined by

$$\begin{aligned} \theta^0_{i,l}(s) =& \{(n,l) : n \in (\min_{i-1}(s), \min_i(s))\,\&\, s(n) = l\}, \text{ extended by 0, and} \\ \theta^1_{i,l}(s) =& \{(n,l) : n \in (\max_i(s), \max_{i-1}(s))\,\&\, s(n) = l\}, \text{ extended by 0.} \end{aligned}$$

In other words, for a given integer n

$$\theta^0_{i,l}(s)(n) = \begin{cases} l & \text{if } n \in (\min_{i-1}(s), \min_i(s)) \text{ and } s(n) = l \\ 0 & \text{otherwise, and} \end{cases}$$

$$\theta^1_{i,l}(s)(n) = \begin{cases} l & \text{if } n \in (\max_i(s), \max_{i-1}(s)) \text{ and } s(n) = l \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $k = 4$, $i = 3$, $l = 2$ and a given sos 4-vector s, $\theta^2_{3,2}(s)$ is the 2-vector such that $(\theta^2_{3,2}(s))(n) = 2$ for every n such that

(a) $s(n) = 2$, and

(b) n is in the interval between $\min_2(s)$ (i.e., the first m such that $s(m) = 2$) and $\min_3(s)$ (i.e., the first m such that $s(m) = 3$), and it is zero otherwise.

For $1 \leq l \leq k$, let

$$\theta^2_l(s) = \{(n,l) : n \in (\min_k(s), \max_k(s))\,\&\, s(n) = l\} \text{ extended by zero.}$$

We illustrate this with another example: For $k = 4$, $l = 3$ and an sos 4-vector s, $\theta^2_3(s)$ is the 3-vector with value $l = 3$ in every element n of the support of s such that

(a) $s(n) = 3$, and
(b) n is in between $\min_4(s)$ and $\max_4(s)$, and 0 otherwise.

By technical convenience, we declare $\theta^0_{i,-1}, \theta^1_{i,-1}$ and θ^2_{-1} the 0 mapping (hence, the equivalence relations associated to them are all equal to FIN^2_k). Also, for $I = \emptyset$, the mappings $\min_I$ and $\max_I$ are simply the 0 functions, i.e., $0(s)(n) = 0$ for all s and n.

REMARK 3.9. (i) Sometimes we will use $\min_i$ or $\max_i$ as a integers instead of i-vectors, i.e., for example $\min_i(s)$ will denote the unique integer n such that $\min_i(s)(n) = i$.
(ii) Also, we can extend the mappings f defined before for FIN_k to all $\mathrm{FIN}_{\le k}$ by setting $\bar{f}(s) = f(s)$, if it is well defined, and $\bar{f}(s) = 0$, if not. For example, for a $(\le k)$-vector s, $\min_i(s)(n) = i$ iff $i \in \mathrm{Range}\ s$ and n is the minimum m such that $s(m) = i$, and $\min_i(s) = 0$ otherwise; and $\theta^0_{i,l}(s)$ will have the same definition, provided that the mappings $\min_{i-1}$ and $\min_i$ are well defined for s, and so on.

Proposition 3.10. *Suppose that l is such that $-1 < l \le i-1$. Then,*

(i) $\sim_{\theta^0_{i,l}} \subseteq \sim_{\min_{i-1}} \cap \sim_{\min_i}$, $\sim_{\theta^1_{i,l}} \subseteq \sim_{\max_i} \cap \sim_{\max_{i-1}}$, *and* $\sim_{\theta^2_l} \subseteq \sim_{\min_k} \cap \sim_{\max_k}$.
(ii) $\sim_{\theta^2_l} \subseteq \sim_{\theta^2_{l+1}}$ *and* $\sim_{\theta^\varepsilon_{i,l}} \subseteq \sim_{\theta^\varepsilon_{i,l+1}}$.

PROOF. We prove the result in (i) for $\theta^0_{i,l}$. The other cases can be shown in a similar way. Suppose that $\theta^0_{i,l}(s) = \theta^0_{i,l}(t)$; we show that $\min_{i-1}(s) = \min_{i-1}(t)$. Let n be such that $\min_{i-1}(t)(n) = i-1$. By symmetry, it suffices to prove that $s(n) = i-1$. So, let r be the unique integer such that $T^{k-C_A(t)(r)}a_r(n) = i-1$. Note that $C_A(t)(r) \ge i-1$. There are two cases to consider:
(a) $C_A(t)(r) = i-1$. Since a_r is an sos, there is some $m \ge n$ such that $T^{k-C_A(t)(r)}a_r(m) = l$, and hence $\theta^0_{i,l}(t)(m) = l$ and $\theta^0_{i,l}(s)(m) = l$. This implies that $C_A(s)(r) = C_A(t)(r)$, and hence $T^{k-C_A(t)(r)}a_r \sqsubseteq s$. Hence, $s(n) = T^{k-C_A(t)(r)}a_r(n) = i-1$.
(b) $C_A(t)(r) > i-1$. Then, $\theta^0_{i,l}$ is well defined for $T^{k-C_A(t)(r)}a_r$, and $\theta^0_{i,l}(T^{k-C_A(t)(r)}a_r) \sqsubseteq \theta^0_{i,l}(t) = \theta^0_{i,l}(t)$, which implies that $T^{k-C_A(t)(r)}a_r \sqsubseteq s$, and again we are done.

Let us now prove the result for θ^2_l in (ii). Suppose that $\theta^2_l(s) = \theta^2_l(t)$, i.e.,

$$\{n \in [\min_k(s), \max_k(s)] \,:\, s(n) = l\} = \{n \in [\min_k(s), \max_k(s)] \,:\, t(n) = l\}.$$

Let $n \in (\min_k(s), \max_k(s))$ be such that $s(n) = l+1$. We show that $t(n) = l+1$. Let r be the unique integer such that $T^{k-C_A(s)(r)}a_r(n) = l+1$. Then, $C_A(s)(r) \ge l+1$, and since a_r is an sos, $T^{k-C_A(s)(r)}a_r^{-1}\{l\} \ne \emptyset$. Moreover,

Claim. $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k(s), \max_k(s)) \ne \emptyset$.

Proof of Claim: Let r_0, r_1 be the unique integers such that $a_{r_0}(\min_k(s)) = a_{r_1}(\max_k(s)) = k$. Observe that $r_0 \le r \le r_1$. There are two cases: If $r_0 < r < r_1$, then we are done since $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap [\min_k(s), \max_k(s)] = (T^{k-C_A(s)(r)}a_r)^{-1}\{l\}$ is non empty.

Suppose that $r_0 = r$ (the case $r_1 = r$ is similar). Then, $C_A(s)(r) = k$, and $\min_k s = \min_k a_r$. So, $(a_r)^{-1}\{l\} \cap (\min_k(a_r), \max_k(s)) \ne \emptyset$, since a_r is an sos, and therefore $\mathrm{Range}\ a_r \restriction (\min_k a_r, \max_k a_r) = \{0, \dots, k\}$. $\square$

Now that for every $m \in (T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k s, \max_k s)$ one has that $t(m) = l$, since $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k s, \max_k s) \subseteq \theta^2_l(t)$. By Proposition 2.3, $C_A(t)(r) = C_A(s)(r)$, and hence $T^{k-C_A(s)(r)}a_r \sqsubseteq t$, which implies that $t(n) = T^{k-C_A(s)(r)}a_r(n) = s(n) = l$.

The second inclusion in (ii) is shown in a similar manner. The details are left to the reader. □

The collection of mappings introduced in Definition 3.8 can be divided into pieces as follows.

Definition 3.11. Let $\mathcal{F}_{\min} = \{\min_1, \dots, \min_k\}$, $\mathcal{F}_{\max} = \{\max_1, \dots, \max_k\}$, $\mathcal{F}_{\mathrm{mid}^\varepsilon} = \{\theta^\varepsilon_{i,l} : i \in \{1, \dots, k\}\ l \in \{1, \dots, i-1\}\}$, for $\varepsilon = 0, 1$, and $\mathcal{F}_{\mathrm{mid}} = \{\theta^2_l : l \in \{1, \dots, k\}\} \cup \{0\}$. Set

$$\mathcal{F} = \mathcal{F}_{\min} \cup \mathcal{F}_{\max} \cup \mathcal{F}_{\mathrm{mid}^0} \cup \mathcal{F}_{\mathrm{mid}^1} \cup \mathcal{F}_{\mathrm{mid}}.$$

Given a k-block sequence A we say that a function $f : \langle A \rangle \to \mathrm{FIN}_{\leq k}$ is a *staircase* function (in A) if it is in the lattice closure of $\mathcal{F}$. An equivalence relation $\sim$ in A is a *staircase* (in A) iff $\sim = \sim_f$ for some staircase mapping f.

Definition 3.12. Let $f, g : \langle A \rangle \to \mathrm{FIN}_k$ be two functions defined on the k-combinatorial subspace defined by A.
(i) We say that f and g are *incompatible*, and we write $f \perp g$, when $f(s) \perp f(s)$ for every $s \in \langle A \rangle$.
(ii) We write $f < g$ to denote that $f(s) < g(s)$ for every $s \in \langle A \rangle$.
(iii) We say that f and g are *equivalent* (in A), and we write $f \equiv g$, when $\sim_f \equiv \sim_g$, i.e., if f and g define the same equivalence relation in A.

Remark 3.13. The family $\mathcal{F}$ is pairwise incompatible, i.e. if $f \neq g$ in $\mathcal{F}$ then $f \perp g$. Also, if $f < g$ then $f \perp g$.

The following makes the notion of staircase relation more explicit.

Proposition 3.14. *Suppose that A is an sos, and suppose that $f : \langle A \rangle \to \mathrm{FIN}_{\leq k}$. Then the following are equivalent: (i) f is staircase.*
(ii) There are $I_\varepsilon \subseteq \{1, \dots, k\}$, $J_\varepsilon \subseteq \{j \in I_\varepsilon : j-1 \in I_\varepsilon\}$, $(l^{(\varepsilon)}_j)_{j \in J_\varepsilon}$ with $l^{(\varepsilon)}_j \leq j-1$ (for $\varepsilon = 0, 1$) and $l^{(2)}_k$ such that

$$f = \min_{I_0} \vee \bigvee_{j \in J_0} \theta^0_{j, l^{(0)}_j} \vee \theta^2_{l^{(2)}_k} \vee \max_{I_1} \vee \bigvee_{j \in J_1} \theta^1_{j, l^{(1)}_j}.$$

We say that $(I_0, J_0, (l^{(0)}_j)_{j \in J_0}, I_1, J_1, (l^{(1)}_j)_{j \in J_1}, l^{(2)}_k)$ are the values *of f.*
(iii) Either $f = 0$ or there is a unique sequence $f_0 < f_1 < \cdots < f_n$, $f_0 \neq 0$ such that $f \equiv \bigvee_{i=0}^n f_i$ in A.

Proof. This decomposition is a direct consequence of the fact that $\mathcal{F}$ is a pairwise incompatible family and the inclusions exposed in Proposition 3.10. □

Proposition 3.15. *Fix a staircase mapping f with decomposition $f = f_0 \cup \cdots \cup f_n$ with $f_0 < \cdots < f_n$ in $\mathcal{F}$, an sos $A = (a_n)_n$ and k-vectors s and t of A. Then*

(i) *$f(s) = f(t)$ if and only if $f_i(s) = f_i(t)$ for every $0 \leq i \leq n$.*
(ii) *$f(s) = f(t)$ iff $f(s \restriction \mathrm{supp}\ t) = f(t)$ and $f(t \restriction \mathrm{supp}\ s) = f(s)$.*[2]

[2]Notice that $s \restriction \mathrm{supp}\ t$ is not necessarily a k-vector, but we can still apply f to it; see Remark 3.9.

(iii) *Suppose that* $s_0, s_1 < t_0, t_1$ *are* $(\le k)$*-vectors of* A *such that* s_0+t_0*,* s_1+t_1 *and* s_0+t_1 *are* k*-vectors. If* $f(s_0+t_0)=f(s_1+t_1)$*, then* $f(s_0+t_0)=f(s_0+t_1)$*.*

Proof. (ii) follows from the fact that $f_i < f_j$ for $i<j$. Let us check (ii) using (i). We may assume that $f\in\mathcal{F}$. There are several cases to consider.
(a) $f=\min_i$. Suppose that $\min_i(s)=\min_i(t)$. Then, $i\in \mathrm{Range}\ s{\upharpoonright}\mathrm{supp}\ t$, and hence $\min_i(s{\upharpoonright}\mathrm{supp}\ t)=\min_i s=\min_i t=\min_i(t{\upharpoonright}\mathrm{supp}\ s)$. Suppose now that $\min_i s<\min_i t$. Then, $\min_i s< \min_i t\le \min_i(t{\upharpoonright}\mathrm{supp}\ s)$. So, $\min_i(t{\upharpoonright}\mathrm{supp}\ s)\neq \min_i s$.
(b) $f=\max_i$ is shown in the same way.
(c) $f=\theta^0_{i,l}$. Suppose that $\theta^0_{i,l}(s)=\theta^0_{i,l}(t)$. Then, by *(a)*, $\min_j s=\min_j t{\upharpoonright}\mathrm{supp}\ s$, and $\min_j t= \min_j s{\upharpoonright}\mathrm{supp}\ t$, where $j=i-1$ or $j=i$. Fix $n\in(\min_{i-1}(s),\min_i(s))$ such that $s(n)=l$. Then, $t(n)=l$, and hence $\theta^0_{i,l}(t{\upharpoonright}s)(n)=l$. Now suppose that $\theta^0_{i,l}(t{\upharpoonright}s)(n)=l$. Then, $t(n)=l$, and hence $s(n)=l$.

Suppose that $\theta^0_{i,l}(s)=\theta^0_{i,l}(t{\upharpoonright}\mathrm{supp}\ s)$ and $\theta^0_{i,l}(t)=\theta^0_{i,l}(s{\upharpoonright}\mathrm{supp}\ t)$. Then, $\min_j(s)=\min_j(t)$ for $j=i-1,i$. Fix n such that $\theta^0_{i,l}(s)(n)=l$. Then, $\theta^0_{i,l}(t{\upharpoonright}\mathrm{supp}\ s)(n)=l$, which implies that $t(n)=l$.
(d) The cases of $f=\theta^1_{i,l}$ and $f=\theta^2_l$ have a similar proof that *(c)*.

Let us prove (iii). To do this, fix s_0,s_1,t_0,t_1 as in the statement, and suppose that $f(s_0+t_0)=f(s_1+t_1)$. Suppose that $f=\min_i$. If $\min_i(s_0+t_0)=\min_i(s_0)$, then clearly $\min_i(s_0+t_0)=\min_i(s_0+t_1)$. If not we have that $\min_i(s_0+t_0)=\min_i(t_0)$, hence by our assumptions $\min_i(s_1+t_1)=\min_i(t_0)$. Since $s_1<t_0$, it follows that $\min_i(s_1+t_1)=\min_i(t_1)$ and we are done. Suppose now that $f=\max_i$. If $\max_i(s_0+t_0)=\max_i(s_0)$, then $\max_i(s_1+t_1)=\max_i(s_1)$ (now using the fact that $t_1>s_0$), and therefore, t_1 is a $(<i)$-vector. So, $\max_i(s_0+t_0)=\max_i(s_0+t_1)$. If $\max_i(s_0+t_0)=\max_i(t_0)$, then $\max_i(s_1+t_1)=\max_i(t_1)$ and we are done. Suppose now that $f=\theta^0_{i,l}$ and suppose that $\theta^0_{i,l}(s_0+t_0)(n)=\theta^0_{i,l}(s_1+t_1)(n)=l$. If $s_1(n)=l$, then $s_0(l)=l$, and hence $(s_0+t_1)(n)=l$. If $t_1(n)=l$, then clearly $(s_0+t_1)(n)=l$. By symmetry, we are done in this case. The cases $f=\theta^0_{i,l}$ and $f=\theta^2_l$ have a similar proof. We leave the details to the reader. □

Proposition 3.16. *Any staircase equivalence relation is canonical.*

Proof. By Proposition 3.15, it suffices to prove the result only for staircases functions $f\in\mathcal{F}$. So, we fix $f\in\mathcal{F}$, set $\sim\,=\,\sim_f$ and consider an equation $p(x_0,\dots,x_n)\sim q(x_0,\dots,x_n)$ where $p(x_0,\dots,x_n)=\sum_{d=0}^n T^{k-m_d}x_d$ and $q(x_0,\dots,x_n)=\sum_{d=0}^n T^{k-u_d}x_d$. Set $p^*=p\circ \mathtt{x}^{-1}$ and $q^*=q\circ\mathtt{x}^{-1}$. So $p^*(d)=m_d$ and $q^*(d)=u_d$ for $d\le n$ and 0 for the rest. Fix two sos's A and B (B can be equal to A), and suppose that $p(a_0,\dots,a_n)\sim_f q(a_0,\dots,a_n)$ for some $(a_0,\dots,a_n)\in[A]^{[n+1]}$. We show that $p(b_0,\dots,b_n)\sim q(b_0,\dots,b_n)$ for every $(b_0,\dots,b_n)\in[B]^{[n+1]}$. There are several cases to consider depending on f.
(a) $f=\min_i$. Let d_0 be the first d such that $m_d\ge i$, and d_1 be the first d such that $u_d\ge i$. Then $\min_i(p(a_0,\dots,a_n))=\min_i(T^{k-m_{d_0}})a_{d_0}$ and $\min_i(q(a_0,\dots,a_n))=\min_i(T^{k-u_{d_1}})a_{d_1}$. Since $\min_i(T^{k-m_{d_0}})a_{d_0}=\min_i(T^{k-u_{d_1}})a_{d_1}$, we have that $d_0=d_1$ (otherwise, $a_{d_0}\perp a_{d_1}$). Hence $m_{d_0}=u_{d_1}$ (because $T^ra\perp T^sa$ if $r\neq s$). So p and q satisfy that for every $d<d_0$, both m_d and u_d are less than i and $m_{d_0}=u_{d_0}=i$. This implies that $\min_i p(b_0,\dots,b_n)=T^{k-m_{d_0}}b_{d_0}=\min_i q(b_0,\dots,b_n)$.

(b) $f = \max_i$ has a similar proof.
(c) $f = \theta^0_{i,l}$. By Proposition 3.10, $\sim_{\theta^0_{i,l}} \subseteq \sim_{\min_{i-1}} \cap \sim_{\min_i}$. Hence $\min_{i-\varepsilon} p(a_0,\dots,a_n) = \min_{i-\varepsilon} q(a_0,\dots,a_n)$ for $\varepsilon = 0,1$. Define, for $\varepsilon = 0,1$, d_ε as the least integer d such that $p^*(d_j) = q^*(d_j) \geq i-1+\varepsilon$. So, $d_0 \leq d_1$ and

$$\theta^0_{i,l} p(a_0,\dots,a_n) = \theta^0_{i,l} \sum_{d=d_0}^{d_1} T^{k-m_d} a_d \tag{7}$$

$$\theta^0_{i,l} q(a_0,\dots,a_n) = \theta^0_{i,l} \sum_{j=d_0}^{d_1} T^{k-u_d} a_d. \tag{8}$$

We see now that for every $d \in [d_0,d_1]$ either m_d and u_d are both less than l or $m_d = u_d$. To do this, suppose that $d \in [d_0,d_1]$ is such that $m_d \geq l$. Then $\theta^0_{i,l} T^{k-m_d} a_d \sqsubseteq \theta^0_{i,l} p(a_0,\dots,a_n) = \theta^0_{i,l} q(a_0,\dots,a_n)$. Since for $d \neq d''$ in $[d_0,d_1]$ one has that $T^{k-u_{d'}} a_{d'} \perp T^{k-m_d} a_d$, it follows that $T^{k-m_d} a_d \sqsubseteq T^{k-u_d} a_d$, and hence $u_d = m_d$.
(d) The cases $f = \theta^1_{i,l}$ and $f = \theta^2_k$ have a similar proof. □

Let us now give some other properties of equations for staircase equivalence relations.

Proposition 3.17. *Suppose that $\sim$ is a staircase equivalence relation with values I_0, J_0, I_1, J_1, $(l^{(0)}_j)_{j \in J_0}$, $(l^{(1)}_j)_{j \in J_1}$ and $l^{(2)}_k$, and suppose that A is an sos.*

- (i) *Let $0 \leq r_0 < r_1 \leq r_2$. If $T^{k-r_0} x_0 + T^{k-r_2} x_1 + x_2 \sim T^{k-r_0} x_0 + x_2$ is true in A, then $r_1 \notin I_0$.*
- (ii) *If $l^2_k = -1$, then the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true in A. If $l^2_k \neq -1$, then for every $0 < l < l^2_k$ the equation $x_0 + T^{k-l} x_1 + x_2 \sim x_0 + x_2$ holds in A.*
- (iii) *Suppose that $i \notin I_0$, and let $j = \max I_0 \cap [1,i]$. Then the equation $T^{k-j} x_0 + T^{k-i} x_1 + x_2 \sim T^{k-j} x_0 + x_2$ is true in A.*
- (iv) *If $l^{(0)}_j = -1$, then the equation $T^{k-(j-1)} x_0 + T^{k-(j-1)} x_1 + x_2 \sim T^{k-(j-1)} x_0 + x_2$ is true in A.*
- (v) *Suppose that $l^{(0)}_j \neq -1$, and let $h < l^{(0)}_j$. Then the equation $T^{k-(j-1)} x_0 + T^{k-h} x_1 + x_2 \sim T^{k-(j-1)} x_0 + x_2$ is true in A.*
- (vi) *Suppose that $p(x_0,\dots,x_n)$ is a $(\leq k)$-term, and suppose that $p(x_0,\dots,x_n) + T^{k-l} x_{n+1} + x_{n+3} \sim p(x_0,\dots,x_n) + T^{k-l} x_{n+2} + x_{n+3}$ holds in A. Then $p(x_0,\dots,x_n) + T^{k-l} x_{n+1} + x_{n+2} \sim p(x_0,\dots,x_n) + x_{n+2}$ also holds.*

The analogous symmetric results are also true.

Proof. We give some of the proofs. The rest are quite similar, and the details are left to the reader. The main idea is to use the decomposition of $f = \bigvee_{i=0}^n f_i$ be the decomposition of f into elements of $\mathcal{F}$ with $f_0 < \cdots < f_n$.
(i): Fix $(a_0,a_1,a_2) \in [A]^{[3]}$. Then $\min_{r_1}(T^{k-r_0} a_0 + T^{k-r_2} a_1 + a_2) = \min_{r_1} T^{k-r_2} a_1$, while $\min_{r_1}(T^{k-r_0} a_0 + a_2) = \min_{r_1}(a_2)$. Hence $\min_{r_1}(T^{k-r_0} a_0 + T^{k-r_2} a_1 + a_2) \neq \min_{r_1}(T^{k-r_0} a_0 + a_2)$.

For the rest of the points (ii) to (vi) one shows that in each case the corresponding equations for $\sim_{f_i}$ hold for every $0 \leq i \leq r$, and then use Proposition 3.15 to conclude that the desired equation also holds. □

Definition 3.18. We call a staircase relation a min-*relation* if its corresponding set $I_1 = \emptyset$, and a max-*relation* if $I_0 = \emptyset$.

Remark 3.19. (i) Proposition 3.10 states that if $l_k^2 \neq -1$, then $k \in I_0 \cap I_1$. Hence if $\sim$ is a min-relation or a max-relation, then $l_k^2 = -1$.
(ii) The equation $x + s \sim x + t$ is true if $\sim$ is a min-relation and the relation $s + x \sim t + x$ is true if $\sim$ is a max-relation.

4. The main Theorem

The next theorem is the main result of this paper.

Theorem 4.1. *For every k and every equivalence relation $\sim$ on FIN_k there is an sos B such that $\sim$ restricted to $\langle B\rangle$ is a staircase equivalence relation.*

Again we use Taylor's result, now to expose the role of equations. Fix an equivalence relation $\sim$ on FIN. A diagonal procedure shows that we can find a block sequence $A = (a_n)_n$ such that for every $i_0, i_1, i_2, i_3, j_0, j_1, j_2, j_3 \in \{0, 1\}$ and every $s, t \in \langle A\rangle$, the equation

$$s + T^{i_0}x_0 + T^{i_1}x_1 + T^{i_2}x_2 + T^{i_3}x_3 \sim t + T^{j_0}x_0 + T^{j_1}x_1 + T^{j_2}x_2 + T^{j_3}x_3 \text{ is decided in } A. \quad (9)$$

For arbitrary k, the corresponding result is stated in Lemma 4.2. We consider the same cases considered in original Taylor's proof:
(a) $x_0 \sim x_1$ holds. Then $\sim$ is $\langle A\rangle^2$ on $\langle A\rangle$: Let $s, t \in \langle A\rangle$, pick $u > s, t$, and hence $s, t \sim u$.
(b) $x_0 \sim x_1$ is false, $x_0 + x_1 \sim x_0$ is true, and $x_0 + x_1 \sim x_1$ is false. Let us check that $\sim$ is $\sim_{\min}$ on $\langle A\rangle$. Fix $s, t \in \langle A\rangle$. Suppose that $s \sim_{\min} t$, and let n be the least integer such that $C_A(s)(n) = 1$. Then $s = a_n + s'$, $t = a_n + t'$, and using the fact that $x_0 + x_1 \sim x_0$ holds, $s, t \sim a_n$. Suppose now that $s \not\sim_{\min} t$, and suppose that $\min(s) < \min(t)$, and pick n as before. Then $s \sim a_n$, $a_n < t$, and $a_n \sim t$, a contradiction.
(c) $x_0 \sim x_1$ is false, $x_0 + x_1 \sim x_0$ is false, and $x_0 + x_1 \sim x_1$ is true. Similar proof that 2. shows that $\sim$ is $\sim_{\min}$ on $\langle A\rangle$.
(d) $x_0 \sim x_1$ is false, $x_0 + x_1 \sim x_0$ and $x_0 + x_1 \sim x_1$ are false, and $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. We show that $\sim$ is $\sim_{\min} \cap \sim_{\max}$ on $\langle A\rangle$. It is rather easy to prove that $\sim_{\min} \cap \sim_{\max} \subseteq \sim$ on $\langle A\rangle$. For the converse, suppose that $\max s \neq \max t$ and $s \sim t$. We may assume that $\max s < \max t$. Let n be the maximal integer m such that $C_A(t)(m) = 1$. Then, $t = t' + a_n$, and hence the equation $s \sim t' + x_0$ holds and hence $t' + x_0 + x_1 \sim t' + x_0$ also holds which implies that $x_0 + x_1 \sim x_0$ holds, a contradiction. Notice that this proves that if $x_0 + x_1 \sim x_0$ is false, then $\sim \subseteq \sim_{\max}$. We assume that $\max s = \max t$ but $\min s \neq \min t$. Suppose that $\min s < \min t$. We show that $s \not\sim t$. Suppose again that $s \sim t$ and work for a contradiction. Let n_0, n_1 be the minimum and the maximum of the support of s in A resp., and let m_0 be the minimum of the support of t i A. Then $s = a_{n_0} + s' + a_{n_1}$, $t = a_{m_0} + t' + a_{n_1}$. Using that the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true, we may assume that $s' = t' = 0$. Since $n_0 < m_1 \leq n_1$, either the equation $x_0 + x_2 \sim x_1 + x_2$ is true or the equation $x_0 + x_1 \sim x_1$ is true. But the first case implies that the equations $x_0 + x_3 \sim x_1 + x_2 + x_3$ and $x_0 + x_3 \sim x_2 + x_3$ hold and hence $x_0 \sim x_0 + x_1$ holds, a contradiction.

(e) $x_0 \sim x_1$, $x_0 + x_1 \sim x_0$, $x_0 + x_1 \sim x_1$, $x_0 + x_1 + x_2 \sim x_0 + x_2$ are false. Then $\sim$ is $=$ on $\langle A\rangle$. Suppose that $s \sim t$, and suppose that $s \neq t$. Since $x_0+x_1 \sim x_0$ is false, then $\max s = \max t$ (see 4. above). Let n be the maximal integer $m < \max s$ such that $C_A(s)(m) \neq C_A(t)(m)$, and without loss of generality we assume that $C_A(s)(n) = 1$ and $C_A(s)(n) = 0$. Then, $s = s' + a_n + s''$, and $t = t' + s''$, with $t' < a_n$. Therefore the equation $s' + x_0 + x_1 \sim t'' + x_1$ holds, which implies that $s' + x_0 + x_1 + x_2, s' + x_0 + x_2 \sim t'' + x_2$ holds, and hence the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true, a contradiction.

For arbitrary k the proof is done by induction on k, making use of several lemmas. From now on we fix an equivalence relation $\sim$ on FIN_k. Our approach is the following. By the pigeonhole principle Theorem 2.4, there is always an sos A who decides a finite class of equations. It turns out that two kind of equations we are interested in are of the form $x_0+s \sim x_0+t$, $s+x_0 \sim t+x_0$ where s and t are $(k-1)$-vectors. The reason is that if they are decided, then we can define naturally the $(k-1)$-equivalence relations

$$s \sim_0 t \text{ iff } s + x_0 \sim t + x_0 \text{ holds,}$$
$$s \sim_1 t \text{ iff } x_0 + s \sim x_0 + t \text{ holds.}$$

and then use the inductive hypothesis to detect both $\sim_0$ and $\sim_1$ as $(k-1)$-staircase equivalence relations. The next thing to do is to interpret $\sim_0$ and $\sim_1$ as k-relations $\sim_0'$ and $\sim_1'$, and then prove that in a suitable restriction $\sim\subseteq\sim_0' \cap \sim_1'$. Finally, a few more equations decided in some sos will force the decomposition $\sim=\sim_0' \cap \sim_1' \cap R$ for a suitable staircase relation R.

Lemma 4.2. *There is some sos $A = (a_n)_n$ such that for every 5-tuples $\vec{i},\vec{j} \in \{0,\dots,k\}^5$, and every $(\leq k)$-vectors s and t of $\langle A\rangle$, the k-equation*

$$s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l \text{ is decided in } A.$$

Proof. We find a fusion sequence $(A_r)_r$ of k-block sequences, $A_r = (a_n^r)_n$ such that for every integer r the equations $s + \sum_{l=0}^4 T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^4 T^{\vec{j}(l)} x_l$ are decided in A_r for every $(\leq k)$-vectors s,t of $\langle (a_i^i)_{i<r}\rangle$ and every $\vec{i},\vec{j} \in \{0,\dots,k\}^5$. Once we have done this, the fusion sequence $A = (a_r^r)_r$ works for our purposes: Fix an equation e, $s + \sum_{l=0}^4 T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^4 T^{\vec{j}(l)} x_l$, and let r be the least integer such that s,t are $(\leq k)$-vectors of $\langle (a_i^i)_{i<r}\rangle$. Then e is decided in A_r, hence it is also decided in A.

We justify the existence of the demanded fusion sequence. Suppose we have already defined $A_r = (a_n^r)_n$. Let $\mathcal{L}$ be the set of all the k-equations of the form

$$s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l$$

where s and t are $(\leq k)$-vectors in $\langle (a_i^i)_{i\leq r}\rangle_{\leq k}$ and $\vec{i},\vec{j} \in \{0,\dots,k\}^5$. Let

$$\Lambda : [(a_n^r)_{n\geq 1}]^{[5]} \to \{0,1\}^{\mathcal{L}}$$

be the finite coloring defined for each $(c_0,\dots,c_4)\in[(a^r_n)_{n\geq 1}]^{[5]}$ and each equation e of the form $s+\sum_{l=0}^4 T^{\vec{i}(l)}x_l\sim t+\sum_{l=0}^4 T^{\vec{j}(l)}x_l\in\mathcal{L}$ by $\Lambda(c_0,\dots,c_4)(e)=0$ iff

$$s+\sum_{l=0}^{4}T^{\vec{i}(l)}c_l\sim t+\sum_{l=0}^{4}T^{\vec{j}(l)}c_l.$$

By Lemma 2.5, there is $A_{r+1}\in[(a^{(r)}_n)_{n\geq 1}]^{[\infty]}$ such that Λ is constant on $[A_{r+1}]^{[5]}$, which is equivalent to all the equations considered above being decided in A_{r+1}. $\square$

4.1. **The inductive step. The relations $\sim_0'$ and $\sim_1'$.** Suppose that Theorem 4.1 holds for $k-1$. Our intention is, of course, to prove the case for k. To do this we first associate two $k-1$-relations to our fixed k-relation $\sim$ as follows.

Lemma 4.3. *There is an sos A and two staircase $k-1$-equivalence relations $\sim_0$ and $\sim_1$ on $\langle A\rangle_{k-1}$ such that for every $s,t\in\langle A\rangle_{k-1}$,*

$$\text{the } k\text{-equation } s+x_0\sim t+x_0 \text{ is true in } A \text{ if and only if } s\sim_0 t, \text{ and} \tag{10}$$

$$\text{the } k\text{-equation } x_0+s\sim x_0+t \text{ is true in } A \text{ if and only if } s\sim_1 t. \tag{11}$$

Moreover $\sim_0$ and $\sim_1$ are such that for any two $(k-1)$-vectors s and t of A,

$$s\sim_0 t \text{ iff the } (k-1)\text{-equation } s+x\sim_0 t+x \text{ holds in } A, \text{ and} \tag{12}$$

$$s\sim_1 t \text{ iff the } (k-1)\text{-equation } x+s\sim_0 x+t \text{ holds in } A. \tag{13}$$

Proof. Let $B=(b_n)_n$ be an sos satisfying Lemma 4.2. Then for $(k-1)$-vectors s and t of B the $(k-1)$-equations $s+x_0\sim t+x_0$ are decided in B. Now define the relation $\sim'$ on $\langle B\rangle_{k-1}$ as follows. For $s,t\in\langle B\rangle_{k-1}$,

$$s\sim' t \text{ iff } s+x_0\sim t+x_0 \text{ holds in } B.$$

It is not difficult to see that $\sim'$ is an equivalence relation. By the inductive hypothesis there is some $(k-1)$-block sequence $B'=(b'_n)_n\in[(Tb_n)_n]^{[\infty]}$ and some canonical equivalence relation $\sim_0$ such that $\sim'$ coincides with $\sim_0$ on B' (since, by Proposition 3.16, all staircase equivalence relations are canonical). The k-block sequence $A=(Sb'_n)_{n\geq 1}$ and the k-equivalence relation $\sim_0$ clearly satisfy (10). We prove assertion (12) for $\sim_0$. To do this, suppose that $s\sim_0 t$. Then the k-equation $s+x_0\sim t+x_0$ holds. Since the equation $s+Tx_0+x_1\sim t+Tx_0+x_1$ is decided, it must be true. It follows that for every k-vector $b>s,t$ we have that $s+Tb\sim_0 t+Tb$. Since $\sim_0$ is canonical, we obtain that the $(k-1)$-equation

$$s+x_0\sim_0 t+x_0 \text{ holds in } A, \tag{14}$$

as desired. Now assume that (14) is true. Fix a $(k-1)$-vector $u>s,t$. Then $s+u\sim_0 t+u$, i.e., the k-equation $s+u+x_0\sim t+u+x_0$ holds. Hence $s+x_0\sim t+x_0$ holds, that is $s\sim_0 t$.

We justify now the existence of a staircase $k-1$-equivalence relation $\sim_1$ and an sos A such that the statements (11) and (13) hold. We can find a fusion sequence $(A_r)_r$, $A_r=(a^r_n)_n$, of k-block sequences of A, and a list $(\sim^n_a)_{a\in\langle(a^i_i)_{i<r}\rangle_k}$ defined on $\langle A_r\rangle_{k-1}$ such that for every $s,t\in\langle A_r\rangle_{k-1}$,

$$a+s\sim a+t \text{ if and only if } s\sim^n_a t.$$

Let $A_\infty = (a_n^n)_n$ be the fusion sequence of $(A_r)_r$. Now for every $a \in \langle A_\infty \rangle$ let $n(a)$ be unique integer unique n such that $a \in \langle (a_i^i)_{i<n} \rangle \setminus \langle (a_i^i)_{i<n-1} \rangle$. Define the finite coloring

$$c : \langle A_\infty \rangle \to \text{canonical equivalence relations on } FIN_{k-1}$$

by $c(a) = \sim_a^{n(a)}$. By Lemma 2.5 there is some $A \in [A_\infty]^{[\infty]}$ in which c is constant, with value $\sim_1$. We check that A and $\sim_1$ satisfy what we want. Fix $a \in \langle A \rangle$ and two $k-1$-vectors s, t of A with $a < s, t$; then $a \in \langle \theta_{n(a)} \rangle$ and s, t are $k-1$-block sequences of $A_{n(a)}$. So, $a + s \sim a + t$ if and only if $s \sim_a^{n(a)} t$ if and only if $s \sim_1 t$. I.e. $x_0 + s \sim x_0 + t$ holds iff $s \sim_1 t$. Notice that in particular all equations $x_0 + s \sim x_0 + t$ are decided in A.

Let us prove now the assertion (13). To do this, fix two $(k-1)$-vectors s, t of A. If $s \sim_1 t$, then $x_0 + s \sim x_0 + t$. Given a $(k-1)$-vector $u < s, t$, choose a k-vector $a < u$ in $\langle (Sb'_n)_{n\geq 0} \rangle$. Then $a + u + s \sim a + u + t$, and this implies that $u + s \sim_1 u + t$; in other words, the $(k-1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Suppose now that the $(k-1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Pick $(k-1)$-vector $u < s, t$. Then the k-equation $x_0 + u + s \sim x_0 + u + t$ is true, and hence also $x_0 + s \sim x_0 + t$ holds (since this equation is decided).

Finally, we justify the existence of the fusion sequence $(A_r)_r$. Suppose we have already defined $A_r = (a_n^r)_n$ fulfilling its corresponding requirements. For every $a \in \langle (a_i^i)_{i<r} \rangle$, put $\sim_a^{n+1} = \sim_a^n$. For every $a \in \langle (a_i^i)_{i\leq r} \rangle \setminus \langle (a_i^i)_{i<r} \rangle$, let R_a be the relation on $\langle (a_n^r)_{n\geq 1} \rangle_{k-1}$ defined by

$$sR_a t \text{ if and only if } a + s \sim a + t.$$

By the inductive hypothesis, we can find some $B \preceq (a_n^r)_{n\geq 1}$ such that for every $a \in \langle (a_i^i)_{i\leq r} \rangle \setminus \langle (a_i^i)_{i<r} \rangle$ the relation R_a is staircase when restricted to B. Then $A_{r+1} = B$ satisfies the requirements. $\square$

Roughly speaking, the assertions (12) and (13) tell that the $(k-1)$-relation $\sim_0$ does not depend on the part of a $(k-1)$-vector before $\min_{k-1}$ and that $\sim_1$ does not depend on the part of a $(k-1)$-vector after $\max_{k-1}$. Indeed (12) and (13) determine the form of $\sim_0$ and $\sim_1$. To express this mathematically we introduce the following useful notation.

Definition 4.4. For $l \leq k$, let $\max_k^l : \mathrm{FIN}_k \to \mathrm{FIN}_k$ be defined by

$$(\max{}_k^l s)(n) = \begin{cases} s(n) & \text{if } n \leq \max_k(s) \text{ and } s(n) \geq l, \\ 0 & \text{otherwise.} \end{cases}$$

In other words $\max_k^l$ is the staircase function with values $I_0 = \{l, \dots, k\}$, $J_0 = \{l+1, \dots, k\}$, for every $j \in J_0$, $l_j^{(0)} = l$, $l_k^{(2)} = l$ and $I_1 = \{k\}$. Symmetrically, we can define $\min_k^l$ by $\min_k^l(n) = s(n)$ iff $n \geq \min_k s$ and $s(n) \geq l$, and 0 otherwise.

Proposition 4.5. *Suppose that* R *is a staircase relation, and suppose that* A *is an sos. The following are equivalent:*

(i) *For every k-vectors s, t of A, one has that $s \,\mathrm{R}\, t$ iff $x + s \,\mathrm{R}\, x + t$ holds in A.*
(ii) *Either* R *is a* max*-relation or there is some* max*-relation* R' *and some* $l \in \{1, \dots, k\}$ *such that* $\mathrm{R} = \mathrm{R}' \cap \max_k^l$.

The analogous result for $s + x \,\mathrm{R}\, t + x$ is also true.

PROOF. Fix a staircase relation R with values $I_\varepsilon, J_\varepsilon, (l_j^{(\varepsilon)})_{j\in J_\varepsilon}$ $(\varepsilon = 0,1)$ and $l_k^{(2)}$ such that for every k-vectors s,t one has that $s \mathrel{\mathrm{R}} t$ iff $x+s \mathrel{\mathrm{R}} x+t$ holds. Suppose that $I_0 \neq \emptyset$, since otherwise R is a max-relation. Let $l = \min I_0$. We show that $I_0 = \{l, l+1, \dots, k\}$, $J_0 = \{l+1,\dots,k\}$, for every $j \in J_0$, $l_j^{(0)} = l$, $l_k^{(2)} = l$ and $k \in I_1$. First we show that $l_k^{(2)} \neq -1$. If not, the equation $x_0 + x_1 + x_2 \mathrel{\mathrm{R}} x_0 + x_2$ is true and hence the equation $x_1 + x_2 \mathrel{\mathrm{R}} x_2$ is true, which implies that $l \notin I_0$, a contradiction. If $l_k^{(2)} > l$, then the equation $x_0 + T^{k-l}x_1 + x_2 \mathrel{\mathrm{R}} x_0 + x_2$ is true and hence the equation $T^{k-l}x_1 + x_2 \mathrel{\mathrm{R}} x_2$ is true, which implies again that $l \notin I_0$. If $l_k^{(2)} < l$, then the equation $T^{k-l_k^{(2)}} x_0 + x_1 \mathrel{\mathrm{R}} x_1$ holds and hence the equation

$$x_0 + T^{k-l_k^{(2)}} x_1 + x_2 \mathrel{\mathrm{R}} x_0 + x_2 \text{ holds,} \tag{15}$$

which contradicts the definition of $l_k^{(2)}$.

We now show that $I_0 = \{l,\dots,k\}$. It is clear that $I_0 \subseteq \{l,\dots,k\}$ since l is the minimum of I_0. We prove the reverse inclusion $\{l,\dots,k\} \subseteq I_0$. Suppose not, and set

$$j = \min\{l,\dots,k\} \setminus I_0.$$

Then the equation $T^{k-j-1}x_0 + T^{k-j}x_1 + x_2 \mathrel{\mathrm{R}} T^{k-j-1}x_0 + x_2$ is true and hence the equation $x_0 + T^{k-j-1}x_1 + T^{k-j}x_2 + x_3 \mathrel{\mathrm{R}} x_0 + T^{k-j-1}x_1 + x_3$ is true, which implies that the equation $x_0 + T^{k-j}x_1 + x_2 \mathrel{\mathrm{R}} x_0 + x_2$ also holds. This contradicts the fact that $j > l$ and that $\sim \subseteq \mathrm{R}_{\theta_l^2}$. Notice that $I_0 = \{l,\dots,k\}$ implies that $J_1 = \{l+1,\dots,k\}$.

We show that $l_j^{(0)} = l$ for all $j \geq l+1$. Suppose that $l_j^{(0)} = -1$. This implies that the equation $T^{k-(j-1)}x_0 + T^{k-(j-1)}x_1 + x_2 \mathrel{\mathrm{R}} T^{k-(j-1)}x_0 + x_2$ holds. Again by adding one variable at the beginning of both terms and using the fact that $j-1 \geq l$ we can arrive at a contradiction to the fact that $l_k^{(2)} = l$. Suppose now that $l_j^{(0)} < l$. Then the equation $T^{k-l_j^{(0)}} x_0 + x_1 \mathrel{\mathrm{R}} x_1$ is true, and adding a variable we arrive at a contradiction. Suppose that $l_j^{(0)} > l$, then the equation

$$T^{k-(j-1)}x_0 + T^{k-l}x_1 + x_2 \mathrel{\mathrm{R}} T^{k-(j-1)}x_0 + x_2 \text{ is true,} \tag{16}$$

which yields a contradiction in the same way as before. It is not difficult to check that the converse and the analogous situation for min are also true. □

Proposition 4.5 and (12) and (13) determine the relations $\sim_0$ and $\sim_1$ as follows.

Corollary 4.6. *The relation $\sim_0$ is either a* min*-relation or there is some $l \leq k-1$ and some* min*-relation R such that $\sim_0 = R \cap \min_{k-1}^l$ and $\sim_1$ is either a* max*-relation or there is some $l \leq k-1$ and some* max*-relation R such that $\sim_1 = R \cap \max_{k-1}^l$.* □

Recall that $\sim_0$ and $\sim_1$ are both staircase equivalence relations of FIN_{k-1}. We now give the proper interpretation of both as k-relations. Suppose that $k > 1$. We know that either $\sim_1$ is a max-relation, or $\sim_1 = \max_{k-1}^l \cap R$, with R a max-relation. Let

$$\sim_1' = \begin{cases} \sim_1 & \text{if } \sim_1 \text{ is a max-relation} \\ \theta_{k,l}^1 \cap R & \text{if } I_0 \neq \emptyset. \end{cases}$$

Notice that in the second case we have that $\max_k \subseteq \sim_1'$. We do the same for $\sim_0$: It is either a min-relation or $\sim_0 = R \cap \min^l_{k-1}$, being R a min-relation. Let

$$\sim_0' = \begin{cases} \sim_0 & \text{if } \sim_0 \text{ is a min-relation} \\ R \cap \theta^0_{k,l} & \text{if } I_1 \neq \emptyset. \end{cases}$$

In this second case we have that $\min_k \subseteq \sim_0'$. For $k = 1$, let $\sim_0' = \sim_1' = \mathrm{FIN}^2_1$.

So, although $\sim_0$ is not a min-relation and $\sim_1$ is not a max-relation, their corresponding interpretations $\sim_0'$ and $\sim_1'$ as k-relations are a min-relation and a max-relation, respectively.

The relations $\sim_0'$ and $\sim_1'$ have similar properties than $\sim_0$ and $\sim_1$.

Proposition 4.7. *Let s and t be $(k-1)$-vectors. Then*

(i) *$s + x \sim_0' t + x$ holds iff $s \sim_0 t$ iff $s + x \sim t + x$ holds.*
(ii) *$x + s \sim_1' x + t$ holds iff $s \sim_1 t$ iff $x + s \sim x + t$ holds.*
(iii) *$s + x_0 + x_1 \sim_0' t + x_0 + x_2$ holds iff $s \sim_0 t$.*
(iv) *$x_0 + x_2 + s \sim_1' x_1 + x_2 + t$ holds iff $s \sim_1 t$.*

Proof. We show the result for $\sim_1$; for $\sim_0$ the proof is similar, and we leave the details to the reader. If $\sim_1$ is a max relation, then there is nothing to prove. Suppose that $\sim_1 = \max^l_{k-1} \cap R$, for some $l \leq k-1$, where R is a max relation. So, $\sim_1' = \theta^{(1)}_{k,l} \cap R$ and we only have to show that

$$s \max^l_{k-1} t \text{ iff the equation } x + s\, \theta^{(1)}_{k,l} x + t \text{ holds,} \tag{17}$$

which is not difficult to check (see Figure below).

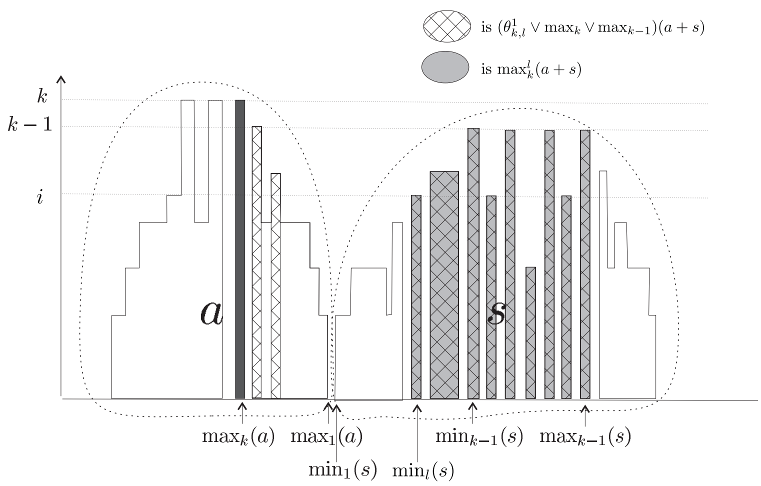

Figure 3. The relation between $\sim_1$ and $\sim_1'$

□

Definition 4.8. Let $D=(d_n)_n$ be a k-block sequence and a k-vector $s=\sum_{n\geq 0} T^{k-C_D(s)(n)}d_n$ of D, and let $n_0=n_0(s)$ and $n_1=n_1(s)$ be respectively the minimal and the maximal elements of the set of integers n such that $C_D(s)(n)=k$. We define the *first part of s in D* as the $(\leq k-1)$-vector $f_D s=\sum_{n<n_0} T^{k-C_D(s)(n)}d_n$, the *middle part of s in D* as the $(\leq k)$-vector $m_D s=\sum_{n\in(n_0,n_1)} T^{k-C_D(s)(n)}d_n$ and the *last part of s in D*, as the $(\leq k-1)$-vector $l_D s=\sum_{n>n_1} T^{k-C_D(s)(n)}d_n$. Using this, we have the decomposition

$$s=f_D s+b_{n_0}+m_D s+b_{n_1}+l_D s.$$

So $f_D s$ is the part of s before the occurrence of $\min_k s$, $m_D s$ is the part of s between $\min_k s$ and $\max_k s$, and $l_D s$ is the part of s after $\max_k s$. All these definitions are local, depending on a fixed sos D.

Let $\mathbb{A}=(a_n)_n$ satisfy both Lemmas 4.2 and 4.3, and let $\mathbb{B}=(b_n)_n$ be defined for every n by $b_n=Ta_{3n}+a_{3n+1}+Ta_{3n+2}$. The role of $\mathbb{B}$ is to guarantee that for every k-vector s of $\mathbb{B}$ the first part $f_{\mathbb{A}}s$ and the last part $l_{\mathbb{A}}s$ are both $(k-1)$-vectors. We need this because $\sim_\varepsilon$ $(\varepsilon=0,1)$ gives information only about $(k-1)$-vectors, since it is a $k-1$-relation.

From now on we work in $\mathbb{B}$, unless we explicitly say the contrary. The following proposition tells us that many equations are decided in $\mathbb{B}$.

Proposition 4.9. *Let $p(x_1,\dots,x_{n-1})$ and $q(x_1,\dots,x_{n-1})$ be $(\leq k-1)$-terms. Then:*

(i) *The equation $x_0+p(x_1,\dots,x_{n-1})\sim x_0+q(x_1,\dots,x_{n-1})$ is decided in $\mathbb{B}$.*

(ii) *The equation $x_0+p(x_1,\dots,x_{n-1})\sim x_0+q(x_1,\dots,x_{n-1})$ holds in $\mathbb{B}$ iff the equation $x_0+p(x_1,\dots,x_{n-1})\sim_1' x_0+q(x_1,\dots,x_{n-1})$ holds in $\mathbb{B}$.*

The analogous results for $\sim_0'$ are also true.

Proof. Fix two $(\leq k-1)$-terms $p=p(x_1,\dots,x_{n-1})$, $q=q(x_1,\dots,x_{n-1})$.
(i) Fix a finite block sequence $(c_0,\dots,c_{n-1})$ in $\mathbb{B}$. Suppose that $c_0+p(c_1,\dots,c_{n-1})\sim c_0+q(c_1,\dots,c_{n-1})$. By definition of $\mathbb{B}$, $c_0=c_0'+c_0''$, where c_0' is a k-vector of $\mathbb{A}$ and c_0'' is a $(k-1)$-vector of $\mathbb{A}$. Hence,

$$c_0''+p(c_1,\dots,c_{n-1})\sim_1 c_0''+q(c_1,\dots,c_{n-1}). \tag{18}$$

Since the relation $\sim_1$ is $(k-1)$-canonical in $\mathbb{A}$, the $(k-1)$-equation

$$x_0+p(x_1,\dots,x_{n-1})\sim_1 x_0+q(x_1,\dots,x_{n-1}) \text{ is true in } \mathbb{A}. \tag{19}$$

Fix $(d_0,\dots,d_{n-1})$ in $\mathbb{B}$, and set $d_0=d_0'+d_0''$. Then,

$$d_0''+p(d_1,\dots,d_{n-1})\sim_1 d_0''+q(d_1,\dots,d_{n-1}), \tag{20}$$

and hence, the equation

$$x_0+p(d_1,\dots,d_{n-1})\sim x_0+q(d_1,\dots,d_{n-1}) \text{ holds in } \mathbb{A}, \tag{21}$$

which implies that $d_0+p(d_1,\dots,d_{n-1})\sim_1 d_0+q(d_1,\dots,d_{n-1})$, as desired.
(ii) Suppose that $x_0+p(x_1,\dots,x_{n-1})\sim x_0+q(x_0,\dots,x_{n-1})$ holds in $\mathbb{B}$. Then for a given block sequence $(c_0,c_1,\dots,c_{n-1})$ in $\mathbb{B}$, the equation

$$x_0+Tc_0+p(c_1,\dots,c_{n-1})\sim x_0+Tc_0+q(c_1,\dots,c_{n-1}) \text{ holds in } \mathbb{B}. \tag{22}$$

By Proposition 4.7, the assertion (22) implies that

$$x_0 + Tc_0 + p(c_1, \dots, c_{n-1}) \sim_1' x_0 + Tc_0 + q(c_1, \dots, c_{n-1}) \text{ holds in } \mathbb{B}. \tag{23}$$

Since $\sim_1'$ is canonical, the equation

$$x_0 + Tx_1 + p(x_2, \dots, x_n) \sim_1' x_0 + Tx_1 + q(x_2, \dots, x_n) \text{ holds in } \mathbb{B}. \tag{24}$$

Therefore the equation $x_0 + p(x_1, \dots, x_{n-1}) \sim_1' x_0 + q(x_1, \dots, x_{n-1})$ holds in $\mathbb{B}$, as desired. □

Proposition 4.10. *Suppose that a, b are k-vectors of $\mathbb{B}$, s, t are $\leq (k-1)$-vectors of $\mathbb{B}$ such that $a < s$, $b < t$ and suppose that $a + s \sim_1' b + t$.*

(i) *If $a, b < s, t$, then $a + s \sim a + t$.*
(ii) *If $l_{\mathbb{A}} a = l_{\mathbb{A}} b = 0$, and $\max_k(a) > \max_k(b)$, then $b + s \sim b + t$.*

The corresponding analogous results for $\sim_0'$ are also true.

Proof. Let us check (i): By point (iv) of Proposition 3.15, we have that $a + s \sim_1' a + t$. By construction of $\mathbb{B}$, $a = a' + a''$ where a' is a k-vector and a'' is a $(k-1)$-vector, both of $\mathbb{A}$. But since the relation is $\sim_1'$ is staircase, it is canonical, and hence the k-equation

$$x_0 + a'' + s \sim_1' x_0 + a'' + t \text{ holds in } \mathbb{A}. \tag{25}$$

It follows from Proposition 4.9 that $a'' + s \sim_1 a'' + t$, and hence, by definition of $\sim_1$, the k-equation

$$x_0 + a'' + s \sim x_0 + a'' + t \text{ holds in } \mathbb{A}. \tag{26}$$

Replacing in (26) x_0 by a', we obtain that $a + s \sim a + t$.
(ii): Since $l_{\mathbb{A}} a = l_{\mathbb{A}} b = 0$, we have that $a + s = a' + a_{n_0} + s$ and $b + t = b' + a_{m_0} + t$. Since $\max_k(a) > \max_k(b)$, it follows that $n_0 > m_0$. This together with the fact that $a + s \sim_1' b + t$ implies that $\max_k \not\subseteq \sim_1'$ and hence, by definition, $\sim_1'$ has to be max-relation. Set $i = \max I_1(\sim_1') < k$. Then the equation

$$p(x_0, \dots, x_r) + T^{k-i'} x_{r+1} \sim_1' q(x_0, \dots, x_r) + T^{k-i'} x_{r+1} \text{ is true}, \tag{27}$$

for every terms p and q, and every $i' \geq i$. Now set

$$t = t' + T^{k-j} a_{n_0} + t''.$$

Notice that t'' is an i-vector, and s is an i'-vector for some $i' \geq i$. By (27),

$$a + s = a' + a_{n_0} + s \sim_1' b' + a_{m_0} + t' + T^{k-j} a_{n_0} + s \sim_1' b' + a_{m_0} + s = b + s. \tag{28}$$

Hence, $b + t \sim_1' b + s$, and since $b < s, t$, 1. implies that $b + s \sim b + t$. □

Our intention is to show that $\sim \subseteq \sim_1'$. To do this, we decompose the relation $\sim_1'$ as the final step of a chain $\sim_1'(1) \subseteq \cdots \subseteq \sim_1'(k) = \sim_1'$ and we prove by induction on j that $\sim \subseteq \sim_1'(j)$.

Definition 4.11. Suppose that R is a max-relation with values $I_0 = \emptyset, I_1, J_1$ and $(l_j^{(1)})_{j \in J_1}$. For every $i \leq k-1$ we define $I_1(i) = I_1 \cap [0, i]$, $J_1(i) = J_1 \cap [0, i]$, and let $\mathrm{R}(i)$ be the staircase equivalence relation on FIN_k with values $I_1(i)$, $J_1(i)$, and $(l_j^{(1)})_{j\in J_1(i)}$. So the relations between $\mathrm{R}(i+1)$ and $\mathrm{R}(i)$ is the following:

$$\mathrm{R}(i+1)=\begin{cases}\mathrm{R}(i) & \text{if } i+1\notin I_1\\ \max_{i+1}\cap\mathrm{R}(i) & \text{if } i+1\in I_1 \text{ and } i\notin I_1\\ \max_{i+1}\cap\mathrm{R}(i)\cap\theta^1_{i+1,l^{(1)}_{i+1}} & \text{if } i+1\in J_1,\end{cases}$$

and $\mathrm{R}=\mathrm{R}(k)$. Observe that each $\mathrm{R}(i)$ is also a staircase equivalence relations on every sos of FIN_i.

Roughly speaking, $\mathrm{R}(i)$ is the staircase equivalence relation whose values are the ones from R which are smaller than i.

Remark 4.12. One has that for a given $i\le k-1$, $s\,\mathrm{R}(i)t$ iff the equation with variable x

$$x+s{\restriction}[\max_i(s),\max_1(s)]\,\mathrm{R}(i)x+t{\restriction}[\max_i(s),\max_1(s)] \text{ holds.} \tag{29}$$

Proposition 4.13. *Suppose that* R *is a* max*-relation of* FIN_k*. Fix* $j'<j<j''$*, and suppose that* s *is a* j'*-vector,* t *is a* $(<j)$*-vector, and* a *is a* j''*-vector such that* $a+s\,\mathrm{R}(j)T^la+t$ *for some* $l>0$*. Then,* $\mathrm{R}(j)=\mathrm{R}(j')$*, and hence* $s''\,\mathrm{R}(j)t''$*.*

Proof. Set $s'=a+s$ and $t'=T^la+t$, and suppose that $s'\,\mathrm{R}(j)t'$. We are going to show that $I_2(j)=I_2(j')$, which will imply that $\mathrm{R}(j)=\mathrm{R}(j')$, as desired. We know that $s'{\restriction}[\max_j(s'),\max_1(s')]\,\mathrm{R}(j)t'{\restriction}[\max_j(s'),\max_1(s')]$. Notice that for every $r\in[j,j')$, $\max_r(s')=\max_r(a)$, hence $\max_r(s')\neq\max_r(t')$, since a and T^la have nothing in common except 0's. This implies that $I_2(j)\subseteq[j',1]$ and hence $I_2(j)=I_2(j')$. □

Lemma 4.14. $\sim\subseteq\sim_1'(j)$*, for every* $j\le k$*. In particular,* $\sim\subseteq\sim_1'$*.*

Proof. The proof is by induction on j. Notice that if $k=1$, then $\sim_1'=\mathrm{FIN}_1^2$ and hence there is nothing to prove. Suppose that $k>1$. Let I_1, J_1 and $(l_j^{(1)})_{j\in J_1}$ be the values of $\sim_1'$.

$\underline{j=1}$: Suppose that $1\in I_1$ (otherwise there is nothing to prove), i.e., $\sim_1'(1)=\sim_{\max_1}$. Suppose that $s\sim t$ but $\max_1(s)<\max_1(t)$, and let n and i be the unique integers such that

$$\max{}_1T^{k-i}a_n=\max{}_1t \text{ and } t=t'+T^{k-i}a_n. \tag{30}$$

So, $s=s'+T^{k-i'}a_n$, for some $i'<i$ and some k-vector s'. The fact that $s\sim t$ implies that the equation $s'+T^{k-i'}x_0\sim t'+T^{k-i}x_0$ holds in $\mathbb{B}$, which implies that the equation $s'+T^{k-i'}(x_0+T^{i'}x_1)\sim t'+T^{k-i}(x_0+T^{i'}x_1)$ is true. Therefore

$$s'+T^{k-i'}x_0+\sim t'+T^{k-i}x_0+T^{k-i+i'}x_1 \text{ holds,} \tag{31}$$

(31) implies that the equation

$$t'+T^{k-i}x_0\sim t'+T^{k-i}x_0+T^{k-i+i'}x_1 \text{ is true,} \tag{32}$$

and hence, also

$$x_0+T^{k-i+i'}x_1\sim x_0 \text{ is true in } \mathbb{B}. \tag{33}$$

But since $j-i+i'<k$, we have that

$$x_0+Tx_1\sim x_0+Tx_1+T^{k-i+i'}x_2 \text{ is true,} \tag{34}$$

and by Proposition 4.9, we have that

$$x_0 + Tx_1 \sim'_1 x_0 + Tx_1 + T^{k-i+i'} x_2 \text{ holds,} \tag{35}$$

which contradicts the fact that $1 \in I_1$.
$\underline{j \curvearrowright j+1}$. Assume that $\sim \subseteq \sim'_1(j)$ and let us conclude that $\sim \subseteq \sim'_1(j+1)$. There are two cases:
(a) $j \notin I_1$: Suppose that $j+1 \in I_1$ (otherwise, there is nothing to prove), and set

$$\beta = \max I_1 \cap [0, j]. \tag{36}$$

Notice that β can be 0. By definition of $\sim'_1$, we know that if $j+1 = k$ belongs to I_1, then $j = k-1$ also belongs to I_1. So, $j+1 < k$. We only need to show that $\sim \subseteq \max_{j+1}$: Suppose that $s \sim t$, and $\max_{j+1} s < \max_{j+1} t$; set $s = s' + T^{k-l} a_n + s''$, $t = t' + T^{k-l'} a_n + t''$, with $l < l'$, $l' \geq j+1$, and $(< (j+1))$-vectors s'' and t''. Observe that in the previous decomposition of s, s' needs to be a k-vector. By the inductive hypothesis,

$$s' + T^{k-l} a_n + s'' \sim'_1(j) t' + T^{k-l'} a_n + t''. \tag{37}$$

Since $\sim'_1$ is a staircase equivalence relation, (iv) of Proposition 3.15 gives that

$$s' + T^{k-l} a_n + s'' \sim'_1(j) s' + T^{k-l} a_n + t'' \ (t'' \text{ can be } 0), \tag{38}$$

which implies that $s' + T^{k-l} a_n + s'' \sim'_1 s' + T^{k-l} a_n + t''$, and hence, by Proposition 4.10, $s' + T^{k-l} a_n + s'' \sim s' + T^{k-l} a_n + t''$. Resuming, we have that

$$s' + T^{k-l} a_n + t'' \sim t' + T^{k-l'} a_n + t'', \tag{39}$$

and hence, the equation

$$s' + T^{k-l} x_0 + T^{k-\alpha} x_1 \sim t' + T^{k-l'} x_0 + T^{k-\alpha} x_1 \text{ holds,} \tag{40}$$

where $j \geq \alpha \geq \beta$ is such that $t'' \in \mathrm{FIN}_\alpha$. Notice that since $j \notin I_1$, and $j \geq \alpha \geq \beta = \max I_1 \cap [0, \dots, j]$, the equation

$$x_0 + T^{k-r} x_1 + T^{k-\alpha} x_2 \sim'_1 x_0 + T^{k-\alpha} x_2 \text{ is true,} \tag{41}$$

for all $r \leq j$. Hence,

$$x_0 + T^{k-r} x_1 + T^{k-\alpha} x_2 \sim x_0 + T^{k-\alpha} x_2 \text{ is true.} \tag{42}$$

There are two now two subcases to consider:
(a.1) $l \leq j$. Then

$$s' + T^{k-\alpha} x_2 \sim s' + T^{k-l} x_1 + T^{k-\alpha} x_2 \sim t' + T^{k-l'} x_1 + T^{k-\alpha} x_2 \text{ is true,} \tag{43}$$

and hence,

$$x_0 + T^{k-l'} x_1 + T^{k-l'} x_2 + T^{k-\alpha} x_3 \sim x_0 + T^{k-l'} x_1 + T^{k-\alpha} x_3 \text{ is true,} \tag{44}$$

which implies that

$$x_0 + T^{k-(j+1)} x_1 + T^{k-\alpha} x_2 \sim x_0 + T^{k-\alpha} x_2 \text{ holds.} \tag{45}$$

By Proposition 4.9,

$$x_0 + T^{k-(j+1)} x_1 + T^{k-\alpha} x_2 \sim'_1 x_0 + T^{k-\alpha} x_2 \text{ holds,} \tag{46}$$

which contradicts the fact that $j+1 \in I_1$.
(a.2) $j+1 \le l < l'$. Then, the equation

$$s' + T^{k-l}(x_0 + T^{l-j}x_1) + T^{k-\alpha}x_2 \sim s' + T^{k-l}x_0 + T^{k-\alpha}x_2 \text{ holds,} \tag{47}$$

and hence,

$$t' + T^{k-l'}x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim t' + T^{k-l'}x_0 + T^{k-\alpha}x_2 \text{ holds,} \tag{48}$$

which implies that

$$x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim x_0 + T^{k-\alpha}x_2 \text{ holds.} \tag{49}$$

Since $i' - i > 0$ the assertion (49) contradicts the fact that $j+1 \in I_1$.
(*b*) $j \in I_1$. We assume that $j+1 \in I_1$ because otherwise there is nothing to prove. Then

$$\sim_1(j+1) = \sim_1(j) \cap \theta^{(1)}_{j+1,l} \cap \max_{j+1},$$

where $l = l^{(1)}_{j+1}$. Suppose that $s \sim t$. By the inductive hypothesis, $s \sim'_1(j)t$, and in particular $\max_j(s) = \max_j(t)$. Let $m_0 = \max\{\max_{j+1}s, \max_{j+1}t\}$. First we show that

$$(s\restriction[m_0, \max_j(s)])^{-1}(l) = (t\restriction[m_0, \max_j(s)])^{-1}(l), \tag{50}$$

i.e., for all $n \in [m_0, \max_j(s)]$, $s(n) = l$ iff $t(n) = l$. Suppose not, and let

$$m_1 = \max\{m \in [m_0, \max_j(s)] \, : \, (s(m) = l \text{ or } t(m) = l) \text{ and } s(m) \neq t(m)\}.$$

Suppose that $s(m_1) = l$, and that $t(m_1) \neq 0$. Let n_1 be the unique integer n such that $T^{k-C_{\mathbb{B}}(n)}a_n(m_1) = s(m_1) = l$, and let $h = C_{\mathbb{B}}(n_1) \geq l$. So, $h' = C_{\mathbb{B}}(n_1) \neq h$, $s = s' + T^{k-h}a_{n_1} + s''$, and $t = t' + T^{k-h'}a_{n_1} + t''$, with s'', t'' both j-vectors. By definition of m_1, the equation

$$x + s'' \sim'_1(j+1)x + t'' \text{ holds,} \tag{51}$$

and hence,

$$x + s'' \sim'_1 x + t'' \text{ and } x + s'' \sim x + t'' \text{ also both hold.} \tag{52}$$

So, $s' + T^{k-h}a_{n_1} + s'' \sim t'' + T^{k-h'}a_{n_1} + s''$, and hence, the equation

$$s' + T^{k-h}x_0 + T^{k-j}x_1 \sim t' + T^{k-h'}x_0 + T^{k-j}x_1 \text{ holds.} \tag{53}$$

There are two subcases to consider:
(b.1) $h > h'$. Since $x_0 + T^{k-r}x_1 + T^{k-j}x_2 \sim'_1 x_0 + T^{k-j}x_2$ is true, the equation $x_0 + T^{k-r}x_1 + T^{k-j}x_2 \sim x_0 + T^{k-j}x_2$ holds for every $r < l$. Since $l + h' - h < l$,

$$s' + T^{k-h}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim s' + T^{k-h}(x_0 + T^{h-l}x_1) + T^{k-j}x_2 \sim \tag{54}$$

$$t' + T^{k-h'}(x_0 + T^{h-l}x_1) + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-(l+h'-h)}x_1 + T^{k-j}x_2 \sim \tag{55}$$

$$\sim t' + T^{k-h'}x_0 + T^{k-j}x_2 \sim s' + T^{k-h}x_0 + T^{k-j}x_2 \text{ hold.} \tag{56}$$

Notice that we have used that $h \geq l$, and so T^{h-l} makes sense. Summarizing, the equation

$$s' + T^{k-h}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim s' + T^{k-h}x_0 + T^{k-j}x_2 \text{ holds,} \tag{57}$$

and hence, the equation

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim'_1 x_0 + T^{k-j}x_2 \text{ holds,} \tag{58}$$

which is a contradiction with the fact that $\sim_1' \subseteq \theta^1_{j+1,l}$.
(b.2) $h < h'$. Then $h' > l$, and repeating the previous argument used for the case $h > h'$, we conclude that the equation

$$t' + T^{k-h'}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-j}x_2 \text{ holds,} \tag{59}$$

and hence,

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim_1' x_0 + T^{k-j}x_2 \text{ holds,} \tag{60}$$

which is a contradiction.

The proof will be finished once we show that $\max_{j+1} s = \max_{j+1} t$. So suppose otherwise, without loss of generality, that $\max_{j+1} s > \max_{j+1} t$. Let $n_1 \in \mathbb{N}$ be such that $\max_{j-1}(s) = \max_{j-1}(T^{k-h}b_{n_1})$, where $h = C_{\mathbb{B}}(n_1) \geq j+1$. Then one has the decomposition $s = s' + T^{k-h}a_{n_1} + s''$, $t = t' + T^{k-h'}a_{n_1} + t''$, where $h' < h$ and s'', t'' are j-vectors. From (50), it follows that

$$x_0 + s'' \sim_1' (j+1)x_0 + t'' \text{ holds,} \tag{61}$$

and hence,

$$s' + T^{k-h}a_{n_1} + t'' \sim t' + T^{k-h'}a_{n_1} + t''. \tag{62}$$

This implies that the equation

$$s' + T^{k-h}x_0 + T^{k-j}x_1 \sim t' + T^{k-h'}x_0 + T^{k-j}x_1 \text{ is true.} \tag{63}$$

Using a similar argument to the above, we arrive at the equation

$$s' + T^{k-h}(x_0 + T^{h-l}x_1) + T^{k-j}x_2 \sim t' + T^{k-h'}(x_0 + T^{h-l}x_1) + T^{k-j}x_2 \text{ is true,} \tag{64}$$

and hence,

$$s' + T^{k-h}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-(l+h'-h)}x_1 + T^{k-j}x_2 \sim \tag{65}$$

$$\sim t' + T^{k-h'}x_0 + T^{k-j}x_2 \sim s' + T^{k-h}x_0 + T^{k-j}x_2 \text{ holds,} \tag{66}$$

which is again a contradiction, since it implies that

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim_1' x_0 + T^{k-j}x_2 \text{ holds.} \tag{67}$$

□

Proposition 4.15. *Suppose that a, b are k-vectors of $\mathbb{B}$, s, t are $(\leq (k-1))$-vectors of $\mathbb{B}$ such that $a < s$ and $b < t$, and suppose that $a + s \sim b + t$.*

- (i) *If $a < t$ and $b < s$, then $a + s \sim a + t$ and hence $a + t \sim b + t$.*
- (ii) *If $a < t$ and $\max_k a < \max_k b$, then $a + s \sim a + t$ and hence $a + t \sim b + t$.*

Proof. (i) is a consequence of Proposition 4.10(1) and Lemma 4.14. Let us prove (ii). To do this, suppose that a, b, s, t are as in the statement. By Lemma 4.14 one has that $a + s \sim_1' b + t$. Since $\max_k(a+s) < \max_k(b+t)$, we have that $\sim_1' = \sim_1$, where $\sim_1$ is a max-relation of FIN_{k-1}. This implies that $s \sim_1 t$, from which the desired result easily follows. □

4.2. **Determining the relation $\sim$.** We already know that $\sim\subseteq\sim_1'$. The following identifies the staircase equivalence relation that will be equal to $\sim$ on $\mathbb{B}$ in terms of which equations hold or not in $\mathbb{B}$. This will conclude the proof of Theorem 4.1.

Theorem 4.16.

(i) *Suppose that* $x_0+T^{k-(l-1)}x_1+x_2\sim x_0+x_2$ *is true, and* $x_0+T^{k-l}x_1+x_2\sim x_0+x_2$ *is false. Then* $\sim=\sim_0'\cap\sim_{\theta_l^2}\cap\sim_1'$.

(ii) *Suppose that* $x_0+x_1+x_2\sim x_0+x_2$ *is true.*

(a) *If* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *and* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *are both false, then* $\sim=\sim_0'\cap\min_k\cap\max_k\cap\sim_1'$.

(b) *If* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *is true, and* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *is false, then* $\sim=\sim_0'\cap\max_k\cap\sim_1'$.

(c) *If* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *is false, and* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *is true, then* $\sim=\sim_0'\cap\min_k\cap\sim_1'$.

(d) *If* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *and* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *are both true, then* $\sim=\sim_0'\cap\sim_1'$.

The proof is done in various steps. (i) is in Corollary 4.20, and (ii.a), (ii.b), (ii.c) and (ii.d) in Corollary 4.25, and Lemmas 4.21, 4.23 and 4.26 respectively. We start with the following proposition that gives one of the inclusions.

Proposition 4.17.

(i) *If the equation* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *is true, then* $\sim_0'\cap\max_k\cap\sim_1'\subseteq\sim$.

(ii) *If the equation* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *is true, then* $\sim_0'\cap\min_k\cap\sim_1'\subseteq\sim$.

(iii) *If the equation* $x_0+T^{k-(l-1)}x_1+x_2\sim x_0+x_2$ *is true, then* $\sim_0'\cap\sim_{\theta_l^2}\cap\sim_1'\subseteq\sim$*, for every* $l\le k$.

(iv) *If the equation* $x_0+x_1+x_2\sim x_0+x_2$ *is true, then* $\sim_0'\cap\min_k\cap\max_k\cap\sim_1'\subseteq\sim$.

(v) *If the equations* $Tx_0+x_1+x_2\sim Tx_0+x_2$ *and* $x_0+x_1+Tx_2\sim x_0+Tx_2$ *are both true, then* $\sim_0'\cap\sim_1'\subseteq\sim$.

Proof. (i): Suppose that the equation $Tx_0+x_1+x_2\sim Tx_0+x_2$ holds. Then, $Tx_0+Tx_1+x_2+x_3\sim Tx_0+x_3$ holds, and the equations $Tx_0+Tx_1+x_2+x_3\sim Tx_0+x_3\sim Tx_0+x_2+x_3$ also hold. This implies that the equation

$$Tx_0+Tx_1+x_2\sim Tx_0+x_2 \text{ is true.} \tag{68}$$

Hence the relation $\sim_0$ is a min-relation, which implies that $\sim_0'$ so is a min-relation. Set $R=\sim_0'\cap\max_k\cap\sim_1'$ and suppose that sRt. Then $\max_k s=\max_k t$. Let n be such that $\max_k s=\max_k=\max_k b_n$. Therefore, $s=s'+a_{3n+1}+s''$ and $t=t'+a_{3n+1}+t''$. It is not difficult to show that the equation

$$Tx_0+x_1+x_2\,R\,Tx_0+x_2 \text{ holds.} \tag{69}$$

So, we may assume that s' and t' are $(k-1)$-vectors of $\mathbb{A}$. Since $s\sim_1' t$, we have that $s'+a_{3n+1}+s''\sim s'+a_{3n+1}+t''$. Since $s\sim_0' t$, we have that $s'+a_{3n+1}+t''\sim_0' t'+a_{3n+1}+t''$, and hence $s'\sim_0 t'$, which implies that $s'+x\sim t'+x$ is true. In particular $s'+a_{3n+1}+t''\sim t'+a_{3n+1}+t''$, i.e., $s\sim t$.

The proofs of (ii), (iii) and (iv) are similar. We leave the details to the reader. Let us check point (v): Fix $s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s$, $t = f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t$ such that sRt, where $R = \sim_0' \cap \sim_1'$. If $m_0 = n_0$, then $sR \cap \min_k t$, and hence we are done by 2. So, suppose that $n_0 < m_0$. Since $\sim_0'$ is a min-relation and $\sim_1'$ is a max-relation, the equations $Tx_0+x_1+x_2RTx_0+x_2$ and $x_0+x_1+Tx_2Rx_0+Tx_2$ are true. Therefore, $sRf_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s$ and $tRf_{\mathbb{A}}t+a_{m_0}+l_{\mathbb{A}}t$. Since $s\sim f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s$ and $t\sim f_{\mathbb{A}}t+a_{m_0}+l_{\mathbb{A}}t$ the proof will be finished if we show that

$$f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t. \tag{70}$$

Since $f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s \sim_0' f_{\mathbb{A}}t+a_{m_0}+l_{\mathbb{A}}t$ and $f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s \sim_1' f_{\mathbb{A}}t+a_{m_0}+l_{\mathbb{A}}t$, by the last point of Proposition 4.10 (for both $\sim_0'$ and $\sim_1'$), we have that

$$f_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t \text{ and } f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t. \tag{71}$$

But $f_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t$, and we are done. □

Lemma 4.18. *If the equation $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is false, then $\sim \subseteq \max_k$.*

Proof. Suppose that $s \sim t$ but $\max_k s > \max_k t$. Set

$$\begin{aligned} s &= f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s \\ t &= f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t, \end{aligned}$$

where $n_1 > m_1$. Set $l_{\mathbb{A}}t = t' + T^{k-i}a_{n_1} + t''$, where $t' < T^{k-i}a_{n_1} < t''$, and $i < k$. By Proposition 4.15

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}a_{n_1} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s, \tag{72}$$

and therefore, the equation

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + x_0 + Tx_1 \text{ holds.} \tag{73}$$

Since $\sim \subseteq \sim_1'$ and $\sim_1'$ is a canonical relation, the $\sim_1'$-equation

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim_1' f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + x_0 + Tx_1 \text{ holds.} \tag{74}$$

Since $\sim_1'$ is a staircase relation, the truth of the last equation implies that $k \notin I_1(\sim_1')$, and hence $\sim_1'$ is a max-relation with $\max(I_1(\sim_1'))$ at most $k-1$. Therefore,

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim_1' f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + Tx_1 \text{ is true,} \tag{75}$$

which implies that

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + Tx_1 \text{ is true.} \tag{76}$$

Hence, the equation

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + Tx_1 \sim_1' f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + x_0 + Tx_1 \text{ holds,} \tag{77}$$

from which we conclude that

$$x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \text{ is true,} \tag{78}$$

a contradiction. □

Lemma 4.19. *Suppose that $x_0+T^{k-(l-1)}x_1+x_2\sim x_0+x_2$ is true but $x_0+T^{k-l}x_1+x_2\sim x_0+x_2$ is false. Then $\sim\subseteq\sim_{\theta_l^2}$. In particular, $\sim\subseteq \min_k\cap\max_k$.*

Proof. Fix l as in the statement. Since we assume that the equation

$$x_0+T^{k-l}x_1+x_2\sim x_0+x_2 \text{ is false,} \tag{79}$$

by Proposition 3.4(1,2), we know that

$$x_0+x_1+Tx_2\sim x_0+Tx_2 \text{ is false.} \tag{80}$$

So, by Lemma 4.18, we obtain that $\sim\subseteq\max_k$. Suppose that $s\sim t$. Take the decomposition

$$\begin{aligned} s=&f_{\mathbb{B}}s+b_{n_0}+m_{\mathbb{B}}s+b_m+l_{\mathbb{B}}s\\ t=&f_{\mathbb{B}}t+b_{n_1}+m_{\mathbb{B}}t+b_m+l_{\mathbb{B}}s, \end{aligned}$$

where we implicitly assume that $l_{\mathbb{B}}s=l_{\mathbb{B}}t$, since $s\sim_1' t$. Observe that showing that $s\theta_l^2t$ is the same that proving that

$$\text{for all } n\in[\min\{n_0,n_1\},m], \text{ either } C_{\mathbb{B}}(s)(n),C_{\mathbb{B}}(t)(n)<l, \text{ or } C_{\mathbb{B}}(s)(n)=C_{\mathbb{B}}(t)(n). \tag{81}$$

Assume on the contrary that (81) is false, and let α be the last $n\in[\min\{n_0,n_1\},m]$ for which

$$\max\{C_{\mathbb{B}}(s)(n),C_{\mathbb{B}}(t)(n)\}\geq l \text{ and } C_{\mathbb{B}}(s)(n)\neq C_{\mathbb{B}}(t)(n). \tag{82}$$

Set $l_0=C_{\mathbb{B}}(s)(\alpha)$, and $l_1=C_{\mathbb{B}}(s)(\alpha)$. Notice that $\alpha<m$. Without loss of generality, we assume that $l_1<l_0$ (the other case has a similar proof). Set

$$\begin{aligned} s'=&\sum_{n<\alpha}T^{k-C_{\mathbb{B}}(s)(n)}b_n\\ t'=&\sum_{n<\alpha}T^{k-C_{\mathbb{B}}(t)(n)}b_n. \end{aligned}$$

Using this notation, we have that the equation

$$s'+T^{k-l_0}x_0+x_1\sim t'+T^{k-l_1}x_0+x_1 \text{ holds.} \tag{83}$$

There are two cases:
$\underline{n_0\leq n_1}$. We first show that in this case $s'+T^{k-l_0}x_0$ is a k-term. If $n_0=n_1$, then $\alpha>n_0$, and hence s' is a k-vector. Suppose that $n_0<n_1$. If $\alpha>n_0$, then s' is a k-term. If $\alpha=n_0$, then $l_0=k$, and clearly $s'+T^{k-k}x_0=s'+x_0$ is a k-term. We consider two subcases:
(a) $l_1<l\leq l_0$. Then, by our assumption that $x_0+T^{k-(l-1)}x_1+x_2\sim x_0+x_2$ holds, we have that

$$s'+T^{k-l_0}x_0+T^{k-l_1}x_1+x_2\sim s'+T^{k-l_0}x_0+x_2 \text{ holds.} \tag{84}$$

By (83),

$$s'+T^{k-l_0}x_0+T^{k-l_1}x_1+x_2\sim t'+T^{k-l_1}x_0+T^{k-l_1}x_1+x_2\sim \tag{85}$$

$$\sim s'+T^{k-l_0}x_0+T^{k-l_0}x_1+x_2 \text{ holds,} \tag{86}$$

which implies that the equation

$$s'+T^{k-l_0}x_0+T^{k-l_0}x_1+x_2\sim s'+T^{k-l_0}x_0+x_2 \text{ holds.} \tag{87}$$

This contradicts the fact that $l_0\geq l$.

(b) $l \leq l_1 < l_0$. Then,

$$s' + T^{k-l_0}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds,} \tag{88}$$

and by (83),

$$s' + T^{k-l_0}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim \tag{89}$$

$$\sim s' + T^{k-l_0}x_0 + T^{k-l}x_1 + x_2 \text{ holds.} \tag{90}$$

Again, this yields a contradiction.
$\underline{n_1 < n_0}$. It can be shown that $t' + T^{k-l_1}x_0$ is a k-term. We consider the same two subcases as above:
(a) $l_1 < l \leq l_0$. Then

$$t' + T^{k-l_1}x_0 + T^{k-l_1}x_1 + x_2 \sim s' + T^{k-l_1}x_0 + x_2 \text{ holds,} \tag{91}$$

and hence,

$$s' + T^{k-l_0}x_0 + T^{k-l_0}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds,} \tag{92}$$

which, by (83), implies that

$$t' + T^{k-l_1}x_0 + T^{k-l_0}x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds,} \tag{93}$$

a contradiction, since $l_0 \geq l$.
(b) $l \leq l_1 < l_0$. Then

$$t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds.} \tag{94}$$

Using that

$$t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim s' + T^{k-l_0}x_0 + T^{k-l}x_1 + x_2 \sim \tag{95}$$

$$\sim t' + T^{k-l_1}x_0 + T^{k-l}x_1 + x_2 \text{ holds,} \tag{96}$$

we arrive at a contradiction. □

Corollary 4.20. *Suppose that* $x_0+T^{k-(l-1)}x_1+x_2 \sim x_0+x_2$ *is true, but* $x_0+T^{k-l}x_1+x_2 \sim x_0+x_2$ *is false. Then,* $\sim\, = \,\sim'_0 \cap \sim_{\theta_l^2} \cap \sim'_1$.

Proof. By Proposition 4.17, $\sim'_0 \cap \sim_{\theta_l^2} \cap \sim'_1 \subseteq \sim$. We only need to show that $\sim\, \subseteq \,\sim'_0$. Suppose that $s \sim t$, and consider the decomposition

$$\begin{aligned} s =& f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}s \\ t =& f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t. \end{aligned}$$

Since $\max_k s = \max_k t$, we have that $m_0 = m_1$, and since $s \sim'_1 t$, by Proposition 4.7(4), we may assume that $l_{\mathbb{A}}s \sim_1 l_{\mathbb{A}}t$. By Lemma 4.19, $s \sim_{\theta_2^l} t$, and using the fact that the equations $x_0 + T^{k-j}x_1 + x_2 \sim x_0 + x_2$ are true for all $j < l$, we may also assume that $n_0 = n_1$ and $m_{\mathbb{A}}s = m_{\mathbb{A}}t$. Therefore, the equation $f_{\mathbb{A}}s + x_0 \sim f_{\mathbb{A}}t + x_0$ holds. By definition of $\sim_0$, we have that $f_{\mathbb{A}}s \sim_0 f_{\mathbb{A}}t$, and by Proposition 4.7(3), $s \sim'_0 t$, as desired. □

Lemma 4.21. *Suppose that* $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ *is true, and* $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ *is false. Then,* $\sim\, = \,\sim'_0 \cap \max_k \cap \sim'_1$.

PROOF. We only need to show that $\sim\ \subseteq\ \sim_0'$. Suppose that $s \sim t$. Consider the following decompositions of s and t

$$\begin{aligned} s =& f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_m + l_{\mathbb{A}}s \\ t =& f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s. \end{aligned}$$

Notice that, since $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is true, we have that $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. Hence, we may assume that $m_{\mathbb{A}}s = m_{\mathbb{A}}t = 0$. Notice also that, since

$$Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \text{ is true,} \tag{97}$$

and since $f_{\mathbb{A}}s$ and $f_{\mathbb{A}}t$ are $(k-1)$-vectors (this is why we use the decompositions of vectors of $\mathbb{B}$ in $\mathbb{A}$), we have that

$$s \sim f_{\mathbb{A}}s + a_{n_2} + l_{\mathbb{A}}s \text{ and } t \sim f_{\mathbb{A}}t + a_{n_2} + l_{\mathbb{A}}s. \tag{98}$$

This implies that $f_{\mathbb{A}}s \sim_0 f_{\mathbb{A}}t$, and, by Proposition 4.7(1,3),

$$s \sim_0' f_{\mathbb{A}}s + a_{n_2} + l_{\mathbb{A}}s \sim_0' f_{\mathbb{A}}t + a_{n_2} + l_{\mathbb{A}}t \sim_0' t, \tag{99}$$

as desired. □

Proposition 4.22. *Suppose that $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ holds, and suppose that $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is false. Then, $\sim\ \subseteq \min_k$.*

PROOF. Suppose that $s \sim t$. Take the decomposition

$$\begin{aligned} s =& f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}s \\ t =& f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t. \end{aligned}$$

Suppose that $n_0 \neq n_1$, and without loss of generality assume that $n_0 < n_1$. Since $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ holds, we have that

$$f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t. \tag{100}$$

By Proposition 4.15(2), we have that

$$f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t, \tag{101}$$

and hence (since $l_{\mathbb{A}}t$ is a $(k-1)$-vector), the equation

$$f_{\mathbb{A}}s + x_0 + Tx_2 \sim f_{\mathbb{A}}t + x_1 + Tx_2 \text{ holds.} \tag{102}$$

This implies that

$$f_{\mathbb{A}}s + x_0 + x_1 + Tx_3 \sim f_{\mathbb{A}}t + x_2 + Tx_3 \sim f_{\mathbb{A}}s + x_1 + Tx_3 \text{ holds,} \tag{103}$$

which implies that the equation

$$f_{\mathbb{A}}s + x_0 + x_2 \sim f_{\mathbb{A}}s + x_2 \text{ is true.} \tag{104}$$

Since $f_{\mathbb{A}}s$ is a $(k-1)$-vector, we have that

$$Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \text{ is true,} \tag{105}$$

a contradiction. □

Lemma 4.23. *Suppose that $x_0+x_1+Tx_2\sim x_0+Tx_2$ is true, and $Tx_0+x_1+x_2\sim Tx_0+x_2$ is false. Then, $\sim=\sim_0'\cap\min_k\cap\sim_1'$.*

Proof. By Proposition 4.17, we have that $\sim_0'\cap\min_k\cap\sim_1'\subseteq\sim$. Let us show that $\sim\subseteq\sim_0'\cap\min_k\cap\sim_1'$. By Proposition 4.22 and Lemma 4.14, we have that $\sim\subseteq\min_k\cap\sim_1'$. So, we only need to show that $\sim\subseteq\sim_0'$. Suppose that $s\sim t$ with

$$\begin{aligned} s=&f_{\mathbb{A}}s+a_{n_0}+m_{\mathbb{A}}s+a_{n_1}+l_{\mathbb{A}}s\\ t=&f_{\mathbb{A}}t+a_{n_0}+m_{\mathbb{A}}t+a_{m_1}+l_{\mathbb{A}}t. \end{aligned}$$

Since the equation $x_0+x_1+Tx_2\sim x_0+Tx_2$ is true, we have that

$$f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s\sim f_{\mathbb{A}}t+a_{n_0}+l_{\mathbb{A}}t, \tag{106}$$

and, by Proposition 4.15,

$$f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s\sim f_{\mathbb{A}}t+a_{n_0}+l_{\mathbb{A}}s, \tag{107}$$

which easily leads to that $s\sim_0' t$. □

Lemma 4.24. *Suppose that $x_0+x_1+x_2\sim x_0+x_2$ is true , and that $x_0+x_1+Tx_2\sim x_0+Tx_2$ and $Tx_0+x_1+x_2\sim Tx_0+x_2$ are both false. Then, $\sim\subseteq\min_k\cap\max_k$.*

Proof. By Lemma 4.18, we know that $\sim\subseteq\max_k$, and by Lemma 4.14, $\sim\subseteq\sim_1'$. So, we only need to show that $\sim\subseteq\min_k$. Suppose that $s\sim t$, set

$$\begin{aligned} s=&f_{\mathbb{A}}s+a_{n_0}+m_{\mathbb{A}}s+a_m+l_{\mathbb{A}}s\\ t=&f_{\mathbb{A}}t+a_{n_1}+m_{\mathbb{A}}t+a_m+l_{\mathbb{A}}t. \end{aligned}$$

Suppose on the contrary that $n_0<n_1$. There are two cases to consider:
$\underline{n_1=m}$. Hence, $n_0<m$ and

$$s\sim f_{\mathbb{A}}s+a_{n_0}+a_m+l_{\mathbb{A}}s \text{ and } t=f_{\mathbb{A}}t+a_m+l_{\mathbb{A}}t. \tag{108}$$

By Proposition 4.15,

$$f_{\mathbb{A}}s+a_{n_0}+a_m+l_{\mathbb{A}}s\sim f_{\mathbb{A}}t+a_m+l_{\mathbb{A}}s, \tag{109}$$

which implies that the equation

$$f_{\mathbb{A}}s+x_0+x_1\sim f_{\mathbb{A}}t+x_1 \text{ is true,} \tag{110}$$

a contradiction, since $f_{\mathbb{A}}s$ is a $(k-1)$-vector.
$\underline{n_1<m}$. Then, by our assumptions, and Proposition 4.15,

$$f_{\mathbb{A}}s+a_{n_0}+a_m+l_{\mathbb{A}}s\sim f_{\mathbb{A}}t+a_{n_1}+a_m+l_{\mathbb{A}}s. \tag{111}$$

Hence, the equation

$$f_{\mathbb{A}}s+x_0+x_2\sim f_{\mathbb{A}}t+x_1+x_2 \text{ is true,} \tag{112}$$

which readily implies that $Tx_0+x_1+x_2\sim Tx_0+x_2$ must be true, a contradiction. □

Corollary 4.25. *Suppose that $x_1+x_2+x_3\sim x_1+x_3$ is true , and that $x_1+x_2+Tx_3\sim x_1+Tx_3$ and $Tx_1+x_2+x_3\sim Tx_1+x_3$ are both false. Then, $\sim=\sim_0'\cap\min_k\cap\max_k\cap\sim_1'$.*

Proof. By Proposition 4.17, $\sim'_0 \cap \min_k \cap \max_k \cap \sim'_1 \subseteq \sim$. Let us show the opposite inclusion. By Lemma 4.24, we have that $\sim \subseteq \min_k \cap \max_k$. It remains to show that $\sim \subseteq \sim'_0$. Suppose that $s \sim t$, where $s = f_{\mathbb{A}}s + a_n + m_{\mathbb{A}}s + a_m + l_{\mathbb{A}}s$ and $t = f_{\mathbb{A}}t + a_n + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s$ (we may assume that $l_{\mathbb{A}}s = l_{\mathbb{A}}t$, since $\max_k(s) = \max_k(t)$). There are two cases: $\underline{n < m}$. Then, $f_{\mathbb{A}}s + a_n + a_m + l_{\mathbb{A}}s \sim f_{\mathbb{A}}t + a_n + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s$ which directly implies that $s \sim'_0 t$. The proof for $\underline{n_0 = m}$ is quite similar. □

Lemma 4.26. *Suppose that $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ are both true. Then, $\sim = \sim'_0 \cap \sim'_1$.*

Proof. It is enough to show that $\sim \subseteq \sim'_0$. Suppose that $s \sim t$, with $s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}s$ and $t = f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t$. We may assume that $s = f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$, and $t = f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t$. W.l.o.g. we assume that $n_0 \le n_1$, and hence, by Proposition 4.15,

$$f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t. \tag{113}$$

Case $n_0 = n_1$. By definition of $\sim'_0$, (113) implies that

$$f_{\mathbb{A}}t + a_{n_0} + l_{\mathbb{A}}t \sim'_0 f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t, \tag{114}$$

but trivially $f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t \sim'_0 f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$, and we are done.
Case $n_0 < n_1$. Then,

$$f_{\mathbb{A}}s + x_0 + Tx_2 \sim f_{\mathbb{A}}t + x_1 + Tx_2 \text{ is true}, \tag{115}$$

which easily yields

$$f_{\mathbb{A}}s + x_1 + Tx_3 \sim f_{\mathbb{A}}t + x_1 + Tx_3 \text{ is true}. \tag{116}$$

This implies that $s \sim'_0 t$. □

Corollary 4.27. *Every equivalence relation on FIN_k is canonical in some sos.* □

This corollary has the following local version.

Corollary 4.28. *For every block sequence A and every equivalence relation $\sim$ on $\langle A\rangle$ there is an sos $B \in [A]^{[\infty]}$ on which $\sim$ is canonical.*

Proof. Fix the canonical isomorphism $\Lambda : \mathrm{FIN}_k \to \langle A\rangle$ (i.e., the extension of $\Theta e_n \mapsto a_n$). It is not difficult to show the following facts:

(i) $B = (b_n)_n$ is an sos iff $FB = (Fb_n)_n$ is an sos.
(ii) For every canonical equivalence relation $\sim_{can}$, every sos B, and $s, t \in \langle B\rangle$, $s \sim_{can} t$ iff $F^{-1}s \sim_{can} F^{-1}t$.

We define $\sim'$ on FIN_k by $s \sim' t$ iff $Fs \sim Ft$. Find a canonical equivalence relation $\sim_{can}$ and an sos B such that $\sim$ and $\sim_{can}$ are the same on $\langle B\rangle$. Let $C = FB$, which is an sos. Then $\sim$ and $\sim_{can}$ are the same in $\langle C\rangle$: $s \sim_{can} t$ iff $F^{-1}s \sim_{can} F^{-1}t$ iff $F^{-1}s \sim' F^{-1}t$ iff $s \sim t$. □

Corollary 4.29. *Every canonical equivalence relation is a staircase equivalence relation.*

Proof. Notice that, since $\sim$ is canonical in A, $\mathbb{A}=A$ works for both Lemmas 4.2 and 4.3. Hence, $\sim$ is a staircase equivalence relation in $\mathbb{B}=(Ta_{3n}+a_{3n+1}+Ta_{3n+2})_n$. Let $\sim'$ be this staircase relation, which is equal to $\sim$ when restricted to $\mathbb{B}$. We show that $\sim$ and $\sim'$ are not only equal in $\mathbb{B}$, but also in A. Fix s and t in A, and take their canonical decompositions in A

$$s=\sum_{n\geq 0}T^{k-C_A(s)(n)}a_n \text{ and } t=\sum_{n\geq 0}T^{k-C_A(t)(n)}a_n.$$

Suppose first that $s\sim t$. Since $\sim$ is canonical, the equation

$$\sum_{n\geq 0}T^{k-C_A(s)(n)}x_n\sim\sum_{n\geq 0}T^{k-C_A(t)(n)}x_n \text{ holds in } A, \tag{117}$$

and hence, also in $\mathbb{B}$, i.e.,

$$\sum_{n\geq 0}T^{k-C_A(s)(n)}x_n\sim'\sum_{n\geq 0}T^{k-C_A(t)(n)}x_n \text{ holds in } \mathbb{B}. \tag{118}$$

But since $\sim'$ is staircase, it is canonical (Proposition 3.16), and hence, equation (118) also holds in A, and in particular, $s\sim' t$.

Suppose now that $s\sim' t$. Since $\sim'$ is canonical in any sos, the equation

$$\sum_{n\geq 0}T^{k-C_A(s)(n)}x_n\sim'\sum_{n\geq 0}T^{k-C_A(t)(n)}x_n \text{ holds in } A, \tag{119}$$

hence, also in $\mathbb{B}$. By definition, $\sim'$ is equal to $\sim$ restricted to $\mathbb{B}$, and hence

$$\sum_{n\geq 0}T^{k-C_A(s)(n)}x_n\sim\sum_{n\geq 0}T^{k-C_A(t)(n)}x_n \text{ holds in } \mathbb{B}. \tag{120}$$

Since $\sim$ is canonical, the equation (120) holds in A, and in particular, $s\sim t$. □

5. Counting

The purpose now is to give an explicit formula for the number t_k of staircase equivalence relations on FIN_k. To do this, recall that $e_n(1)=\sum_{j=0}^n\frac{1}{j!}$ is the exponential sum-function and that $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. Recall also that $\Gamma(n,1)=(n-1)!e^{-1}e_{n-1}(1)$ for every integer n.

Let $\mathcal{A}_k$, $\mathcal{B}_k$ be the set of min-relations and max-relations respectively, and set $a_k=|\mathcal{A}_k|$ and $b_k=|\mathcal{B}_k|$. Let $\mathcal{C}_k\subseteq\mathcal{A}_k$ be the set of min-relations R such that $k\notin I_0(R)$, and let $\mathcal{D}_k\subseteq\mathcal{B}_k$ be the set of max-relations R such that $k\notin I_1(R)$. Set $c_k=|\mathcal{C}_k|$ and $d_k=|\mathcal{D}_k|$. Notice that

(i) $c_k=a_{k-1}$,
(ii) $\mathcal{A}_k=\mathcal{A}_{k-1}\cup\{R\cap\sim_{\min_k}\,:\,R\in\mathcal{C}_{k-1}\}\cup\{R\cap\sim_{\min_k}\cap\sim_{\theta^0_{k,l}}\,:\,l=-1 \text{ or } l=1,\dots,k-1,\ R\in\mathcal{A}_{k-1}\setminus\mathcal{C}_{k-1}\}$. So, $a_k=a_{k-1}+c_{k-1}+k(a_{k-1}-c_{k-1})$.

Hence,

$$a_k=(k+1)a_{k-1}-(k-1)a_{k-2},\ a_0=1,\ a_1=2. \tag{121}$$

By standard methods, we conclude that

$$a_k=\frac{e\ (1+k)\ k!\Gamma(1+k,1)}{\Gamma(2+k)}=k!e_k(1). \tag{122}$$

Now let $\mathcal{T}_k$ be the set of staircase equivalence relations of FIN_k and $t_k = |\mathcal{T}_k|$. Then,

$$\mathcal{T}_k = (\{R \cap S \, : \, R \in \mathcal{A}_k, S \in \mathcal{B}_k\} \setminus \{R \cap S \, : \, R \in \mathcal{A}_k \setminus \mathcal{C}_k, S \in \mathcal{B}_k \setminus \mathcal{D}_k\}) \tag{123}$$

$$\cup \{R \cap S \cap \sim_{\theta_l^2} \, : \, R \in \mathcal{A}_k \setminus \mathcal{C}_k, S \in \mathcal{B}_k \setminus \mathcal{D}_k, l = -1 \text{ or } l = 1, \dots, k\}. \tag{124}$$

Hence,

$$t_k = a_k^2 - (a_k - c_k)^2 + (k+1)(a_k - c_k)^2 = k(a_k - a_{k-1})^2 + a_k^2 \tag{125}$$

and from (122) and (125), we obtain that

$$t_k = (k! e_k(1))^2 + k\left(k! e_k(1) - (k-1)! e_{k-1}(1)\right)^2, \tag{126}$$

or, equivalently,

$$t_k = e^2 \left[k \left[\Gamma(k,1) - \Gamma(k+1,1) \right]^2 + \Gamma(k+1,1)^2 \right]. \tag{127}$$

This is a table with the first few values of t_k:

k	0	1	2	3	4	5	6
t_k	1	5	43	619	13829	446881	19790815

Remark 5.1. Let us say that a canonical equivalence relation R is *linked free* iff $I_0(R)$ and $I_1(R)$ have no consecutive members and $k \notin I_0(R) \cap I_1(R)$. The number l_k of linked free canonical equivalence relations of FIN_k is the Fibonacci number F_{2k+2} for $2k+2$, since F_{l+2} is the number of subsets of $\{1, 2, \dots, l\}$ with no consecutive elements, and since R is linked free iff the set $I_0(R) \cup \{2k+1-i \, : \, i \in I_1(R)\} \subseteq \{1, 2, \dots, 2k\}$ has no consecutive numbers.

6. The finite version

Theorem 6.1. *For every m there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \dots, e_n \rangle$ there is some sos $(a_0, \dots, a_{m-1}) \preceq (e_0, \dots, e_n)$ such that $\sim$ is a staircase equivalence relation in $\langle a_0, \dots, a_{m-1} \rangle$.*

Proof. Suppose not. Then, there is some m such that for every n there is some equivalence relation $\sim_n$ on $\langle e_0, \dots, e_n \rangle$ which is not a staircase relation when restricted to any sos $(a_0, \dots, a_{m-1})$ of $(e_i)_{i=0}^n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and define the equivalence relation $\sim$ on FIN_k by

$$s \sim t \text{ if and only if } \{n \, : \, s R_n t\} \in \mathcal{U},$$

where $R_n = \sim_n \cup (\mathrm{FIN}_k)^2$ is an equivalence relation on FIN_k. It is easy to see that $\sim$ is an equivalence relation. By Theorem 4.1, there is some sos $A = (a_n)_n$ on which $\sim$ is a staircase equivalence relation, say $\sim_{can}$. Choose n large enough such that:

(i) $(a_0, \dots, a_{m-1}) \preceq (e_i)_{i=0}^n$
(ii) For $s, t \in \langle a_0, \dots, a_{m-1} \rangle$ one has that $s \sim t$ iff $s \sim_n t$.

This can be done as follows: For every pair $s, t \in \langle a_0, \dots, a_{m-1} \rangle$, let

$$A_{s,t} = \begin{cases} \{n \, : \, s \sim_n t\} \in \mathcal{U} & \text{if } s \sim t \\ \{n \, : \, s \not\sim_n t\} \in \mathcal{U} & \text{if } s \not\sim t. \end{cases} \tag{128}$$

Let $n = \min \bigcap_{s,t \in \langle a_0, \dots, a_{m-1} \rangle} A_{s,t}$. Then $\sim_n$ is $\sim$ restricted to $(a_0, \dots, a_{m-1})$, and hence is a staircase equivalence relation, a contradiction. $\square$

Corollary 6.2. *For every m there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \dots, e_n\rangle$ there is some sos $(a_0, \dots, a_{m-1}) \preceq (e_0, \dots, e_n)$ such that $\sim$ is a canonical equivalence relation on $\langle a_0, \dots, a_{m-1}\rangle$.* □

Corollary 6.3. *For every m there is some $n = n(m)$ such that for every $(b_0, \dots, b_n)$ and every equivalence relation $\sim$ on $\langle b_0, \dots, b_n\rangle$ there is some sos $(a_0, \dots, a_{m-1}) \preceq (b_0, \dots, b_n)$ such that $\sim$ is a staircase equivalence relation when restricted to $\langle a_0, \dots, a_{m-1}\rangle$.*

Proof. Let $n = n(m)$ be given by Theorem 6.1. Fix $b_0, \dots, b_n$, and an equivalence relation $\sim$. Let F be the canonical isomorphism between $\langle e_0, \dots, e_n\rangle$ and $\langle b_0, \dots, b_n\rangle$. Define $\sim'$ on $\langle e_i\rangle_{i=1}^n$ via F, i.e., $s \sim' t$ if and only if $F(s) \sim F(t)$. Fix an sos $(c_i)_{i=0}^{m-1} \preceq (e_i)_{i=1}^n$ and a staircase equivalence relation $\sim_{can}$ such that $s \sim' t$ if and only if $s \sim_{can} t$, for every $s, t \in \langle c_i\rangle_{i=0}^{m-1}$. Let $b_i = Fc_i$, for every $i = 0, \dots, m-1$. Then $(b_0, \dots, b_m)$ is an sos since sos are preserved under isomorphisms, and $\sim_{can}$ is well defined on $(b_0, \dots, b_{m-1})$. Since $\sim_{can}$ is staircase one has that

$$s \sim_{can} t \text{ if and only if } F^{-1}s \sim_{can} F^{-1}t$$

for every $s, t \in \langle b_0, \dots, b_{m-1}\rangle$. Hence,

$$s \sim_{can} t \text{ iff } F^{-1}s \sim_{can} F^{-1}t \text{ iff } F^{-1}s \sim' F^{-1}t \text{ iff } s \sim t.$$

Therefore $\sim_{can}$ and $\sim$ coincide on $\langle b_0, \dots, b_{m-1}\rangle$. □

Definition 6.4. We say that a staircase relation $\sim$ is *symmetric* iff $I_1(\sim) = I_0(\sim) = I$, $J_1(\sim) = J_0(\sim) = J$ and $l_j^{(0)} = l_j^{(1)}$ for every $j \in J$.

Corollary 6.5. *For every m there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \dots, e_n\rangle$ there are* disjointly supported *sos's $a_0, \dots, a_{m-1} \in \langle (e_i)_{i=0}^n\rangle$ such that $\sim$ is a symmetric staircase relation in $\langle a_0, \dots, a_{m-1}\rangle$.*

Before we give the proof of this, let us observe that if $a_0, \dots, a_{m-1}$ are disjointly supported k-vectors then the mapping $e_i \mapsto a_i$ extends to a lattice-isomorphism from $\langle (e_i)_{i=0}^{m-1} \to \langle (a_i)_{i=0}^{m-1}\rangle$ that preserves the operation T.

Proof. Fix an integer m. Let n be given by Theorem 6.1 when applied to $2m$. Suppose that $\sim$ is an equivalence relation on $\langle (e_i)_{i=0}^n\rangle$. Then there is some sos $(b_i)_{i=0}^{2m-1}$ such that $\sim$ is a staircase relation when restricted to $\langle (b_i)_{i=0}^{2m-1}\rangle$. Let $a_i = b_i + b_{2m-i-1}$ for every $0 \le i \le m-1$. A typical vector $b \in \langle (a_i)_{i=0}^{m-1}\rangle$ is of the form

$$b = \sum_{i=0}^{m-1} T^{k-r_i} b_i + \sum_{i=0}^{m-1} T^{k-r_i} b_{2m-i-1}.$$

Let $s, t \in \langle (a_i)_{i=0}^{m-1}\rangle$. Then one has that

$$\min_i(s) = \min_i(t) \text{ iff } \max_i(s) = \max_i(t), \text{ and} \tag{129}$$

$$\theta_{i,l}^0(s) = \theta_{i,l}^0(t) \text{ iff } \theta_{i,l}^1(s) = \theta_{i,l}^1(t). \tag{130}$$

Let $(I_0, J_0, (l_j^{(0)})_{j\in J_0}, I_1, J_1, (l_j^{(1)})_{j\in J_1}, l_k^{(2)})$ be the values of $\sim$ when restricted to $\langle (b_i)_{i=0}^{2m-1}\rangle$. Using (129) and (130) it follows that our fixed relation $\sim$ is when restricted to $\langle (a_i)_{i=0}^{m-1}\rangle$ a symmetric staircase relation with values

$$(I_0 \cup I_1, J_0 \cup J_1, (l_j)_{j\in J_0\cup J_1}, I_0 \cup I_1, J_0 \cup J_1, (l_j)_{j\in J_0\cup J_1}, l_k^{(2)})$$

and where for each $j \in J_0 \cup J_1$

$$l_j = \begin{cases} \min\{l_j^{(0)}, l_j^{(1)}\} & \text{if } j \in J_0 \cap J_1 \\ l_j^{(0)} & \text{if } j \in J_0 \setminus J_1 \\ l_j^{(1)} & \text{if } j \in J_1 \setminus J_0. \end{cases}$$

□

Remark 6.6. (1) Prömel and Voigt were the firsts to observe in [3] the Corollary 6.5 for FIN. We thank the referee for pointing us out this.
(2) Let $\mathcal{S}_k$ be the set of symmetric staircase relations of FIN_k, and set $s_k = |\mathcal{S}_k|$. Using the notation from the Section 5 one has that

$$\mathcal{S}_k = \mathcal{C}_k \cup \{\sim \cap \sim_{\theta_l^2} \,:\, \sim\in \mathcal{A}_k \setminus \mathcal{C}_k, \text{ and } l = -1, 1, \dots, k\}.$$

Hence

$$s_k = c_k + (a_k - c_k)(k+1) = a_{k-1} + (k+1)(a_k - a_{k-1}) = (k+1)!e_k(1) - k!e_{k-1}(1).$$

7. Canonical relations and continuous maps on PS_{c_0}

Our result on equivalence relations on FIN_k gives some consequences about equivalence relations on PS_{c_0}. Let us start with some natural definitions.

For a fixed $\delta > 0$, let k be the first integer such that $1/(1+\delta)^{k-1} < \delta$, and set $\delta_i = (1+\delta)^{i-k}$, for $0 \le i \le k$. For $0 \le i \le k+1$, let

$$\gamma_i(\delta) = \begin{cases} \frac{\delta_{i-1}+\delta_i}{2} = \frac{\varepsilon^{k-i}(\varepsilon+1)}{2} & \text{if } 1 \le i \le k \\ 0 & \text{if } i = 0 \\ \delta_k = 1 & \text{if } i = k+1 \end{cases}$$

and for $0 \le i \le k$, let

$$I_i^{(\delta)} = \begin{cases} [\gamma_i(\delta), \gamma_{i+1}(\delta)) & \text{if } 0 \le i < k \\ [\gamma_k(\delta), \gamma_{k+1}(\delta) = 1] & \text{if } i = k. \end{cases}$$

We have then that $\delta_i \in I_i^{(\delta)}$ for every $0 \le i \le k$, and that $[0,1] = \bigcup_{i=0}^k I_i^{(\delta)}$, a disjoint union.

For $x = (x_m)_m \in PB_{c_0}$ and $n \in \mathbb{N}$, let $\Gamma_n^{(\delta)}(x)$ be the unique $0 \le i \le k$ such that $x_n \in I_i^{(\delta)}$, and define $\Gamma_\delta : PB_{c_0} \to \mathrm{FIN}_{\le k}$ by $\Gamma_\delta(x) = (\Gamma_n^{(\delta)}(x))_n$. Notice that $\Gamma_\delta(PS_{c_0}) \subseteq \mathrm{FIN}_k$. A vector $x \in PS_{c_0}$ is called a *δ-sos* iff $\Gamma_\delta x$ is an sos. A block sequence $(x_n)_n$ of vectors of PS_{c_0} is called a *δ-sos* iff every $x \in PS_X$ is a δ-sos. The next proposition is not difficult to prove.

Proposition 7.1. *Fix $\rho \in [0,1]$, $x, y \in PB_{c_0}$, and a k-vector s of FIN_k. Let i be the unique integer such that $\rho \in I_i^{(\delta)}$. Then,*

(i) *$\Gamma_\delta(x+y) = \Gamma_\delta(x) + \Gamma_\delta(y)$ and $\Gamma_\delta(\rho e_n) = T^{k-i}\Gamma_\delta(e_n) = T^{k-i}(\Theta_\delta^{-1} e_n)$.*

(ii) $\Gamma_\delta(\rho\Theta_\delta^{-1}x) = T^{k-i}\Gamma_\delta(\Theta_\delta^{-1}x)$. *It follows that if* $(a_n)_n$ *is an sos k-block sequence, then* $(\Theta_\delta^{-1}a_n)_n$ *is a δ-sos.* □

Definition 7.2. Given a staircase mapping f of FIN_k, we consider the following two extensions to an arbitrary δ-sos $X = (x_n)_n$. The fist one is $f^{(0)} : PS_X \to \mathrm{FIN}_{\leq k}$, closing the following diagram:

$$\begin{tikzcd} PS_X \arrow[r, "\Gamma_\delta"] \arrow[dr, "f^{(0)}"'] & \langle(\Gamma_\delta x_n)_n\rangle \arrow[d, "f"] \\ & \mathrm{FIN}_{\leq k} \end{tikzcd}$$

The second one is $f^{(1)} : PS_X \to PB_{c_0}$, defined by $f^{(1)}(x)(n) = x(n)$ iff $f^{(0)}x(n) \neq 0$.

Proposition 7.3. *Fix a staircase* f*, and some* δ*-sos* X*.*

(i) $(f \odot g)^{(i)} = f^{(i)} \odot g^{(i)}$, *for* $i = 0, 1$ *and* $\odot$ *equal to* $\vee$ *or* $\wedge$.
(ii) $f^{(1)}$ *is a Baire class 1 function.*
(iii) *If* $f^{(1)}x = f^{(1)}y$, *then* $f^{(0)}x = f^{(0)}y$ *for every* $x, y \in PS_X$.
(iv) $\|\Theta_\delta^{-1}f^{(0)}x - f^{(1)}x\| \leq \delta$ *for every* $x \in PS_X$.
(v) *For every k-vector* $a \in \langle(\Gamma_\delta x_n)_n\rangle$, $f^{(1)}\Theta_\delta^{-1}a = f^{(0)}\Theta_\delta^{-1}a = fa$. *Therefore* $f^{(1)}\Theta_\delta^{-1}a = f^{(1)}\Theta_\delta^{-1}b$ *iff* $f^{(0)}\Theta_\delta^{-1}a = f^{(0)}\Theta_\delta^{-1}b$, *for every k-vectors* $a, b \in \langle(\Gamma_\delta x_n)_n\rangle$.
(vi) *For every* $x \in PS_X$ *there is some k-vector* $\bar{x}$ *such that* $\|x - f^{(0)}\Theta_\delta^{-1}\bar{x}\| \leq \delta$ *and* $f^{(0)}x = f^{(0)}\Theta_\delta^{-1}\bar{x}$.

Proof. (i) is not difficult to check. Let us show (ii). To do this, suppose that f is a staircase mapping. Then f is in the algebraic closure of $\mathcal{F}$ (see Definition 3.11), i.e., there is a finite list $f_0, \dots, f_n \in \mathcal{F}$ such that $f = f_0 \odot_0 f_1 \odot_1 f_2 \odot_2 \cdots \odot_{n-1} f_n$, where $\odot_i$ is either $\vee$ or $\wedge$ for every $i = 0, \dots, n-1$. By point (i) one has that $f^{(1)} = f_0^{(1)} \odot_0 f_1^{(1)} \odot_1 f_2^{(1)} \odot_2 \cdots \odot_{n-1} f_n^{(1)}$. Since for every point $x \in PS_X$ the support of $f_i^{(1)}(x)$ is finite, we may assume that $f \in \mathcal{F}$. We give the proof for the case $f = \min_i$. The other cases can be shown in a similar way. For $l > 0$ we define the following perturbations of the intervals $I_i^{(\delta)}$, let

$$I_{i,l}^{(\delta)} = \begin{cases} (\gamma_i(\delta) - \frac{1}{l}, \gamma_{i+1}(\delta)) & \text{if } i < k \\ (\gamma_k(\delta) - \frac{1}{l}, 1] & \text{if } i = k. \end{cases}$$

These are open intervals of PS_{c_0}. For each l, let $f_l : PS_X \to PB_X$ be defined for $n \in \mathbb{N}$ as follows,

$$f_l(x)(n) = \begin{cases} x(n) & \text{if } x(n) \in I_{i,l}^{(\delta)} \text{ and for all } m < n \ x(n) \in [0, \gamma_i(\delta)) \\ 0 & \text{if not.} \end{cases}$$

Let us see that f_l is continuous, and that $f_l \to_l f$. Suppose that $x_r \to_r x$, with $x_r, x \in PS_X$. Let n be the unique integer such that $f_l(x)(n) = x(n) > 0$, i.e., $x(n) \in I_{i,l}^{(\delta)}$ and $x(m) \in [0, \gamma_i(\delta))$ for every $m < n$. Since both sets are open, there must be some r' such that $x_{r''}(n) \in I_{i,l}^{(\delta)}$ and

$x_{r''}(m) \in [0, \gamma_i(\delta))$, for every $r'' > r'$ and every $m < n$. Therefore, for all $r'' > r'$, $f_l(x_{r''}) = f_l x$. Let us check now that $f_l \to f$. Fix x, and we show that $f_l(x) \to f(x)$. Again, Let n be the unique integer such that $f_l(x)(n) = x(n) > 0$. Let l' be such that $x(m) \in [0, \gamma_i(\delta) - 1/l)$ for every $m < n$. Then $f_{l''}x(m) = 0$ and $f_{l''}x(n) = x(n)$, for every $l'' \geq l'$ and every $m < n$. Also, $f_{l''}x(m) = 0$ for every $m > n$. All this implies that $f_{l''}(x) = f(x)$.

The rest of the points (iii)-(vi) are not difficult to prove. We leave the details to the reader. □

For an equivalence relation R, and $x \in PS_{c_0}$, the R-*equivalence class* of x is denoted by $[x]_R$.

Proposition 7.4. *Fix $\delta > 0$, a staircase equivalence relation R_f, and a k-block sequence $A = (a_n)_n$, where $k = k(\delta)$. Set $X = (x_n = \Theta_\delta^{-1} a_n)_n$ and $R = R_{f^{(1)}}$.*

(i) *For every $x \in PS_X$ there is a k-vector $\bar{x}$ of A such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}]_R)_\delta$.*

(ii) *For every $x, y \in PS_X$, if $(x, y) \in R$, then $(x, z) \in R$, for every $x \wedge y \leq_L z \leq_L x \vee y$.*

Proof. To prove (i), fix $x \in PS_X$, and let $\bar{x}$ be a k-vector of A such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $f^{(0)}x = f^{(0)}\Theta_\delta^{-1}\bar{x}$. Set $x' = \Theta_\delta^{-1}\bar{x}$. We show that $[x]_R \subseteq ([x']_R)_\delta$. Suppose that $y \in [x]_R \cap PS_X$. Then $f^{(1)}x = f^{(1)}y$, and hence $f^{(0)}y = f^{(0)}x = f^{(0)}x'$. Let $\bar{y}$ be a k-vector of A such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f^{(0)}y = f^{(0)}\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f^{(0)}x' = f^{(0)}y'$, which implies that $f^{(1)}x' = f^{(1)}y'$, i.e., $y' \in [x']_R$ and hence $y \in ([x']_R)_\delta$.

(ii): By Proposition 3.15, we may assume that $f \in \mathcal{F}$. Again, we give a proof for the case $f = \min_i$, since the other cases can be shown in a similar way. Suppose that $(x, y) \in R_{f^{(1)}}$, and fix $z \in PS_X$ with $x \wedge y \leq_L z \leq_L x \vee y$. Let n be the unique integer such that $f^{(1)}x(n) = x(n) = y(n) = f^{(1)}y(n) > 0$. Then $x(m), y(m) \in [0, \gamma_i(\delta))$ for every $m < n$. Therefore, $z(n) = x(n) = y(n)$ and $z(m) \in [0, \gamma_i(\delta))$ for every $m < n$. This implies that $f^{(1)}(z) = f^{(1)}(x)$. □

Definition 7.5. A δ-*staircase equivalence relation* is $R_{f^{(1)}}$ for some staircase f.

The next result is the interpretation of Theorem 4.1 in terms of equivalence relations on PS_X.

Proposition 7.6. *Let R be an equivalence relation on PS_X. Then for every $\delta > 0$ there is some δ-sos X and some δ-staircase equivalence relation $\widetilde{R}$ such that:*

(i) *R and $\widetilde{R}$ coincide in an ε-net of PS_X for some $\varepsilon < \delta$.*

(ii) *For every $\widetilde{R}$-class α on PS_X there is a R-class β on PS_X such that $\alpha \subseteq \beta_\delta$.*

Proof. Fix δ, and let $k = k(\delta)$. Define $\bar{R}$ on FIN_k via Θ_δ. Then there is some sos k-block sequence $A = (a_n)_n$ and some staircase equivalence relation R_f such that $\bar{R}$ and R_f coincide on $\langle A \rangle$. Set $\widetilde{R} = R_{f^{(1)}}$ and $X = (x_n)_n$, where $x_n = \Theta_\delta^{-1} a_n$ for every n.

(i): For $\varepsilon = (1+\delta)^{k-1}$, $N = \Theta_\delta^{-1}(\langle (a_n)_n \rangle)$ is a ε-net of $PS(X)$ satisfying our requirements.

(ii): For a fixed $x \in PS_X$ choose some k-vector $\bar{x}$ of A such that $\|x - x'\| \leq \delta$ and $f^{(0)}x = f^{(0)}x'$, where $x' = \Theta_\delta^{-1}\bar{x}$. We show that $[x]_{\widetilde{R}} \subseteq ([x']_R)_\delta$. Suppose that $y \in PS_X$ is such that $f^{(1)}x = f^{(1)}y$. Pick some k-vector $\bar{y}$ of A such that $\|y - y'\| \leq \delta$ and $f^{(0)}y = f^{(0)}y'$ where $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f^{(0)}x = f^{(0)}y$ and hence $f^{(0)}x' = f^{(0)}y'$, which implies that $f^{(1)}x' = f^{(1)}y'$. Therefore, $y' \in [x']_R$. □

In the case of equivalence relations with some additional properties, we have the following stronger result.

Proposition 7.7. *Fix $\delta, \gamma > 0$, set $k = k(\delta)$, and suppose that R is an equivalence relation on PS_{c_0} such that*

(i) *for every $x, y \in PS_{c_0}$ and every $z \in PS_{c_0}$ with $x \wedge y \leq_L z \leq_L x \vee y$, if $(x, y) \in R$, then $(x, z) \in R$, and*

(ii) *for every sos k-block sequence $B = (b_n)_n$ and every $x \in PS_{(\Theta_\delta^{-1} b_n)_n}$ there is some k-vector $\bar{x}$ of B such that $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}]_R)_\gamma$.*

Then, there is some δ-sos X and some δ-staircase equivalence relation $\widetilde{R}$ such that
(a) *for every R-equivalent classes α in PS_X, there is a $\widetilde{R}$-equivalent class β in PS_X such that $\alpha \subseteq \beta_{\delta+\gamma}$, and*
(b) *for every $\widetilde{R}$-equivalence class β there is a R-equivalence class α such that $\beta \subseteq (\alpha)_\delta$.*

Proof. Define $\bar{R}$ on FIN_k via Θ_δ. Then, there is some sos $A = (a_n)$ and some staircase equivalence relation R_f such that $\bar{R}$ is R_f on $\langle A \rangle$. Let $\widetilde{R} = R_{f^{(1)}}$, and $X = (x_n)_n$, where $x_n = \Theta_\delta^{-1} a_n$ for every n. *(b)* is shown in Proposition 7.6. Let us show *(a)*. Fix $x \in PS_X$, and choose a k-vector $\bar{x}$ of A such that $[x]_R \subseteq ([x']_R)_\gamma$ where $x' = \Theta_\delta^{-1}\bar{x}$. Let us show that $[x']_R \subseteq ([x']_{\widetilde{R}})_\delta$ on PS_X. Fix $y \in [x']_R$. Then, there is some k-vector $\bar{y}$ of A such that $x' \wedge y \leq_L y' \leq_L x' \vee y$ and $\|y - y'\| \leq \delta$, where $y' = \Theta_\delta^{-1}\bar{y}$. Hence, $y' \in [x']_R$, and therefore, $y' \in [x']_{R'}$. ◻

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